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ABSTRACT

The accuracy of the Markov chain Monte Carlo procedure, Gibbs sampling, was considered for estimation of item and ability parameters of the one-parameter logistic model. Four data sets were analyzed to evaluate the Gibbs sampling procedure. Data sets were also analyzed using methods of conditional maximum likelihood, marginal maximum likelihood, and joint maximum likelihood. Two different ability estimation methods, maximum likelihood and expected a posteriori, were employed under the marginal maximum likelihood estimation of item parameters. Item parameter estimates from Gibbs sampling were similar to those obtained from the expected a posteriori method. (Contains 12 figures, 23 tables, and 60 references.) (Author)

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An Evaluation of a Markov Chain Monte Carlo Method for the Rasch Model

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Abstract

The accuracy of the Markov chain Monte Carlo procedure, Gibbs sampling, was considered for estimation of item and ability parameters of the one-parameter logistic model. Four data sets were analyzed to evaluate the Gibbs sampling procedure. Data sets were also analyzed using methods of conditional maximum likelihood, marginal maximum likelihood, and joint maximum likelihood. Two different ability estimation methods, maximum likelihood and expected a posteriori, were employed under the marginal maximum likelihood estimation of item parameters. Item parameter estimates from the four methods were almost identical. Ability estimates from Gibbs sampling were similar to those obtained from the expected a posteriori method.

Index terms: Bayesian inference, conditional maximum likelihood, Gibbs sampling, item response theory, joint maximum likelihood, Markov chain Monte Carlo, marginal maximum likelihood, Rasch model.

Introduction

Some problems in statistical inference require integration over possibly high-dimensional probability distributions in order to estimate model parameters of interest or to obtain characteristics of model parameters. One such problem is estimation of item and ability parameters in the context of item response theory (IRT). Except for certain rather simple problems with highly structured frameworks (e.g., an exponential family together with conjugate priors in Bayesian inference), the required integrations may not be analytically feasible. Many efficient numerical approximation strategies have been recently developed for complicated integrations. In this paper, we examine the accuracy of one of the efficient numerical approximation strategies, a Markov Chain Monte Carlo (MCMC) method, for estimation of IRT item and ability parameters. We focus on the accuracy of a particular MCMC procedure, Gibbs sampling (Geman & Geman, 1984), for estimation of item and ability parameters under the one-parameter logistic (1PL) model (Rasch, 1960/1980).

A number of ways exist for implementing the MCMC methods. For a review, refer to Bernardo and Smith (1994), Carlin and Louis (1996), and Gelman, Carlin, Stern, and Rubin (1995). Metropolis and Ulam (1949), Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller (1953), and Hasting (1970) present a general framework within which Gibbs sampling (Geman & Geman, 1984) can be considered as a special case. In this regard, Gelfand and Smith (1990) discuss several different Monte Carlo-based approaches, including Gibbs sampling, for calculating marginal densities. Gilks, Richardson, and Spiegelhalter (1996) contains a recent survey of applications of Gibbs sampling. Basically Gibbs sampling is applicable for obtaining parameter estimates from the complicated joint posterior distribution in Bayesian estimation under IRT (e.g., Mislevy, 1986; Swaminathan & Gifford, 1982, 1985, 1986; Tsutakawa & Lin, 1986).

Albert (1992) applied Gibbs sampling in the context of IRT to estimate item parameters for the two-parameter normal ogive model and compared these estimates with those obtained using maximum likelihood estimation. Baker (1998) has also investigated item parameter recovery characteristics of Albert's Gibbs sampling method for item parameter estimation via a simulation study. Patz and Junker (1997) developed a MCMC method based on the Metropolis-Hasting algorithm and presented an illustration using the two-parameter logistic model.

MCMC computer programs in IRT have been developed largely only for specific

applications. For example, Albert (1992) used a computer program written in MATLAB (The MathWorks, Inc., 1996). Baker (1998) developed a specialized FORTRAN version of Albert's Gibbs sampling program to estimate item parameters of the two parameter normal ogive model. Patz and Junker (1997) developed an S-PLUS code (MathSoft, Inc., 1995). Spiegelhalter, Thomas, Best, and Gilks (1997) have also developed a general Gibbs sampling computer program BUGS for Bayesian estimation, using the adaptive rejection sampling algorithm (Gilks & Wild, 1992). The computer program BUGS requires specification of the complete conditional distributions.

For the Rasch model (Rasch, 1960/1980; Fischer & Molenaar, 1995; Wright & Stone, 1979) many estimation methods can be used to obtain item and ability parameter estimates (Molenaar, 1995; Hoijtnik & Boomsma, 1995). Item and person parameters can be estimated jointly by maximizing the joint likelihood function (i.e., JML, Wright & Stone, 1979). Conditional maximum likelihood (CML) seems to be the standard estimation method under the Rasch model for estimation of item parameters (e.g., Molenaar, 1995). Also, Marginal maximum likelihood (MML) estimation using the expectation and maximization algorithm can be used to obtain item parameter estimates (Thissen, 1982). In addition, joint Bayesian estimation and marginal Bayesian estimation can be employed to obtain parameter estimates under the Rasch model (e.g., Swaminathan & Gifford, 1982). The Gibbs sampling procedure approaches the estimation of item and ability parameters using the joint posterior distribution rather than the marginal distribution. Even so, all methods should yield comparable item parameter estimates, especially when comparable priors are used or when ignorance or locally-uniform priors are used. This paper was designed to investigate this issue using the 1PL model. Specifically, item and ability estimates from the methods of Gibbs sampling, CML, MML, and JML, were examined and compared.

Theoretical Framework

Joint Estimation Procedures

Consider binary responses to a test with n items by each of N examinees. A response of examinee i to item j is represented by a random variable Y_{ij}, where $i = 1(1)N$ and $j = 1(1)n$. The probability of a correct response of examinee i to item j is given by $P(Y_{ij} = 1|\theta_i, \xi_j) = P_{ij}$ and the probability of an incorrect response is given by $P(Y_{ij} = 0|\theta_i, \xi_j) = 1 - P_{ij} = Q_{ij}$, where θ_i is ability and ξ_j is the item parameter or possibly the vector of item parameters.

For examinee i, there is an observed vector of dichotomously scored item responses of length n, $Y_i = (Y_{i1}, \ldots, Y_{in})'$. Under the assumption of conditional independence, the probability of Y_i given θ_i and the vector of all item parameters, $\xi = (\xi_1, \ldots, \xi_n)'$, is

$$p(Y_i|\theta_i, \xi) = \prod_{j=1}^{n} P_{ij}^{Y_{ij}} Q_{ij}^{1-Y_{ij}}. \tag{1}$$

The probability of obtaining the $N \times n$ response matrix Y is given by

$$p(Y|\theta, \xi) = \prod_{i=1}^{N} \prod_{j=1}^{n} P_j(\theta_i)^{Y_{ij}} Q_j(\theta_i)^{1-Y_{ij}} = l(\theta, \xi|Y), \tag{2}$$

where $\theta = (\theta_1, \ldots, \theta_N)'$. Note that $l(\theta, \xi|Y)$ can be regarded as a joint function of θ and ξ given the data Y. Wright and Stone (1979) describe the joint estimation of θ and ξ (cf. Birnbaum, 1968; Lord, 1980, 1986). In implementation of JML, the item parameter estimation part for maximizing $l(\xi|Y, \hat{\theta})$ and the ability parameter estimation part for maximizing $l(\theta|Y, \hat{\xi})$ are iterated until a stable set of maximum likelihood estimates of item and ability parameters is obtained.

Extending the idea of joint maximization, Swaminathan and Gifford (1982, 1985, 1986) suggested that θ and ξ can be estimated by joint maximization with respect to the parameters of the posterior density

$$p(\theta, \xi|Y) = \frac{p(Y|\theta, \xi)p(\theta, \xi)}{p(Y)} \propto l(\theta, \xi|Y)p(\theta, \xi), \tag{3}$$

where $\propto$ denotes proportionality and $p(\theta, \xi)$ is the prior density of the parameters θ and ξ. This procedure is called joint Bayesian estimation. A prior distribution represents what is known about unknown parameters before the data are obtained. Prior knowledge or even relative ignorance can be represented by such a distribution. Under the assumption that priors of θ and ξ are independently distributed with probability density functions $p(\theta)$ and $p(\xi)$, the item parameter estimation part maximizing $l(\xi|Y, \hat{\theta})p(\xi)$, and the ability parameter estimation part maximizing $l(\theta|Y, \hat{\xi})p(\theta)$ are iterated to obtain the Bayes modal estimates of item and ability parameters.

Conditional Maximum Likelihood

Andersen (1970, 1972) showed that consistent estimates of item parameters can be obtained using the conditional estimation procedure. The conditional estimation procedure is based on the availability of sufficient statistics for the ability parameters. Under the Rasch model,

the number correct score, $R_i = \sum_j Y_{ij}$, is the sufficient statistics for θ_i and, consequently, R is the sufficient statistics for θ. For a given examinee with θ_i, the conditional probability of Y_i given R_i can be written as

$$\frac{p(Y_i|\theta_i, \xi)}{p(R_i|\theta_i, \xi)} = p(Y_i|R_i, \xi) \tag{4}$$

which does not contain θ_i. Hence, the entire likelihood can be expressed in terms of R instead of θ, that is,

$$l(\xi|Y, R). \tag{5}$$

The CML estimates of item parameters can be obtained by maximizing the conditional likelihood function without any reference to the ability parameters. The ability parameters are estimated separately under CML, in general, using the maximum likelihood method. The conditional likelihood function involves computing elementary symmetric functions (see Baker & Harwell, 1996).

Marginal Estimation Procedures

In marginal solutions, ability will be integrated out from either the likelihood function or the posterior distribution. The marginal probability of obtaining the response vector Y_i for examinee i sampled from a given population is

$$p(Y_i|\xi) = \int p(Y_i|\theta_i, \xi)p(\theta_i)d\theta_i, \tag{6}$$

where $p(\theta_i)$ is the population distribution of θ_i. Without loss of generality, we can assume that the θ_i are independent and identically distributed as standard normal, $\theta_i \sim N(0, 1)$. This assumption may be relaxed as the ability distribution can also be empirically characterized (Bock & Aitkin, 1981). The marginal probability of Y_i can be approximated with any specified degree of precision by Gaussian quadrature formulas (Stroud & Secrest, 1966).

The marginal probability of obtaining the $N \times n$ response matrix Y is given by

$$p(Y|\xi) = \prod_{i=1}^{N} p(Y_i|\xi) = l(\xi|Y), \tag{7}$$

where $l(\xi|Y)$ can be regarded as a function of ξ given the data Y. In MML, this marginal likelihood is maximized to obtain maximum likelihood estimates of item parameters (Bock & Aitkin, 1980; Thissen, 1982). Ability parameters are estimated after obtaining the item parameter estimates assuming the estimates are the true parameter values.

Bayes' theorem tells us that the marginal posterior probability distribution for ξ given Y is proportional to the product of the marginal likelihood for ξ given Y and the prior distribution of ξ. That is,

$$p(\xi|Y) = \frac{p(Y|\xi)p(\xi)}{p(Y)} \propto l(\xi|Y)p(\xi). \tag{8}$$

The marginal likelihood function represents the information obtained about ξ from the data. In this way, the data modify our prior knowledge of ξ. In marginal Bayesian estimation of item parameters, the marginal posterior is maximized to obtain Bayes modal estimates of item parameters (Mislevy, 1986).

Gibbs Sampling

The main feature of MCMC methods is to obtain a sample of parameter values from the posterior density (Tanner, 1996). The sample of parameter values then can be used to estimate some functions or moments (e.g., mean and variance) of the posterior density of the parameter of interest. In comparison, in the above IRT estimation procedures via JML, CML, or MML, the task is to obtain modes of the likelihood function or of the posterior distribution.

The Gibbs sampling algorithm is as follows (Gelfand & Smith, 1990; Tanner, 1996). First, instead of using θ and ξ, let ω be a vector of parameters with k elements. Suppose that the full or complete conditional distributions, $p(\omega_i|\omega_j, Y)$, where $i = 1(1)k$ and $j \neq i$, are available for sampling. That is, samples may be generated by some method given values of the appropriate conditioning random variables. Then given an arbitrary set of starting values, $\omega_1^{(0)}, \ldots, \omega_k^{(0)}$, the algorithm proceeds as in Figure 1. The vectors $\omega^{(0)}, \ldots, \omega^{(t)}, \ldots$ are a realization of a Markov chain with a transition probability from $\omega^{(t)}$ to $\omega^{(t+1)}$ given by

$$p(\omega^{(t)}, \omega^{(t+1)}) = \prod_{l=1}^{k} p(\omega_l^{(t+1)}|\omega_j^{(t)}, j > l, \omega_j^{(t+1)}, j < l, Y). \tag{9}$$

Insert Figure 1 about here

The joint distribution of $\omega^{(t)}$ converges geometrically to the posterior distribution $p(\omega|Y)$ as $t \to \infty$ (Geman & Geman, 1984; Bernardo & Smith, 1994). In particular, $\omega_i^{(t)}$ tends to be distributed as a random quantity whose density is $p(\omega_i|Y)$. Now suppose that there exist

m replications of the t iterations. For large t, the replicates $\omega_{i1}^{(t)}, \ldots, \omega_{im}^{(t)}$ are approximately a random sample from $p(\omega_i|Y)$. If we make m reasonably large, then an estimate, $\hat{p}(\omega_i|Y)$, can be obtained either as a kernel density estimate derived from the replicates or as

$$\hat{p}(\omega_i|Y) = \frac{1}{m}\sum_{l=1}^{m} p(\omega_i|\omega_{jl}^{(t)}, j \neq i, Y). \tag{10}$$

In the context of IRT, Gibbs sampling tries to obtain or sample sets of parameters from the joint posterior density $p(\theta, \xi|Y)$. Inferences with regard to parameters can then be made using the sampled parameters. Note that inference for both θ and ξ can be made from the Gibbs sampling procedure.

Steps of Gibbs Sampling

Gibbs sampling uses the following four basic steps (cf. Spiegelhalter, Best, Gilks, & Inskip, 1996):

1. Full conditional distributions and sampling methods for unobserved parameters must be specified.

2. Starting values must be provided.

3. Output must be monitored.

4. Summary statistics (e.g., estimates and standard errors) for quantities of interest must be calculated.

Discussion of the four steps involved are presented in detail below using four data sets (i.e., Examples 1 to 4), especially in Example 1. In addition, comparisons with the results from CML, MML, and JML as implemented in the computer programs, PML (Molenaar, 1990), BILOG (Mislevy & Bock, 1990) and BIGSCALE (Wright, Linacre, & Schultz, 1989), are presented. The four data sets analyzed in Examples 1 to 4 represent different calibration situations under the Rasch model, ranging from an extremely small number of items/examinees to a relatively large number of items/examinees.

Example 1

Data

The first example is presented using the familiar Law School Admission Test Section 6 (LSAT6) data from Bock and Lieberman (1970) (see also Andersen, 1980; Bock & Aitkin,

1981). The LSAT6 data are given in Table 1. Model parameters were estimated by Gibbs sampling using the computer program BUGS (Spiegelhalter et al., 1997). These same LSAT6 data have been analyzed under the 1PL model and under the two-parameter normal ogive (i.e., probit) model in Spiegelhalter, Thomas, Best, and Gilks (1996). Spiegelhalter, Thomas, et al. (1996) also compared the BUGS results with those from Bock and Aitkin (1981).

Insert Table 1 about here

Model Specifications

The model specifications are used as input to the BUGS computer program. In the LSAT6 data set, the item responses Y_{ij} are independent, conditional on their parameters P_{ij}. For examinee i and item j, each P_{ij} is a function of the ability parameter θ_i, the location parameter β_j, and the slope parameter α under the 1PL (cf. Thissen, 1982). The θ_i are assumed to be independently drawn from a standard normal distribution for scaling purposes. Figure 2 is adopted from Spiegelhalter, Thomas, et al. (1996) and shows a directed acyclic graph (see Lauritzen, Dawid, Larsen, & Leimer, 1990; Whittaker, 1990; Spiegelhalter, Dawid, Lauritzen, & Cowell, 1993) based on these assumptions. It is only possible to proceed by following the directions of the arrows. Each variable or quantity in the model appears as a node in the graph, and directed links correspond to direct dependencies as specified above. The solid arrow denotes the probabilistic dependency, while dashed arrows indicate functional or deterministic relationships. The rectangle designates observed data, and circles represent unknown quantities. The model can be seen as directed because each link between nodes is represented as an arrow. The model can also be seen as acyclic because it is impossible to return to a node after leaving.

Insert Figure 2 about here

It may be helpful to use the following definitions: Let v be a node in the graph, and V be the set of all nodes. A parent of v is defined as any node with an arrow extending from it and pointing to v, and a descendant of v is defined as any node on a direct path beginning from v. For identifying parents and descendants, deterministic links should be combined so that, for example, the parent of Y_{ij} is P_{ij}. It is assumed in Figure 2 for any node v, if we

know the value of its parents, then no other nodes would be informative concerning v except descendants of v.

Lauritzen et al. (1990) indicated that, in a full probability model, the directed acyclic graph model is equivalent to assuming that the joint distribution of all the random quantities is fully specified in terms of the conditional distribution of each node given its parents. That is,

$$P(V) = \prod_{v \in V} P(v|\text{parents}[v]), \tag{11}$$

where $P(\cdot)$ denotes a probability distribution. This factorization not only allows extremely complex models to be built up from local components, but also provides an efficient basis for the implementation of MCMC methods (Spiegelhalter; Best, et al., 1996).

Gibbs sampling via the BUGS computer program works by iteratively drawing samples from the full conditional distributions of unobserved nodes in Figure 2 using the adaptive rejection sampling algorithm (Gilks, 1996; Gilks & Wild, 1992). For any node v, the remaining nodes are denoted by $V - v$. It follows that the full conditional distribution, $P(v|V-v)$, has the form

$$\begin{aligned} P(v|V-v) &\propto P(v, V-v) \\ &\propto P(v|\text{parent}[v]) \prod_{w \in \text{children}[v]} P(w|\text{parents}[w]). \end{aligned} \tag{12}$$

The proportionality constant, which is a function of the remaining nodes, ensures that the distribution is a probability function that integrates to unity.

To analyze the LSAT6 data, we begin by specifying the forms of the parent and child relationships in Figure 2. Under the 1PL model, the probability that examinee i responds correctly to item j is assumed to follow a logistic function

$$P_{ij} = \frac{\exp(\alpha\theta_i - \beta_j)}{1 + \exp(\alpha\theta_i - \beta_j)} = \frac{1}{1 + \exp[-(\alpha\theta_i - \beta_j)]}. \tag{13}$$

For scaling purposes, we may use the form

$$\theta_i' - b_j = \alpha\theta_i - \beta_j, \tag{14}$$

where θ_i' is the usual Rasch ability parameter and b_j is the Rasch item difficulty parameter defined as $\theta_i' = \alpha\theta_i - \bar{\beta}$ and $b_j = \beta_j - \bar{\beta}$, where $\bar{\beta}$ is the mean of the location parameters, $\bar{\beta} = \sum_j \beta_j / n$. Since Y_{ij} are Bernoulli with parameter P_{ij}, we can define

$$Y_{ij} \sim \text{Bernoulli}(P_{ij}) \tag{15}$$

and

$$\text{logit}(P_{ij}) = \alpha\theta_i - \beta_j. \tag{16}$$

To complete the specification of a full probability model in for the BUGS computer program, prior distributions of the nodes without parents (i.e., θ_i, β_j, and α) also need to be specified. We can define these priors in several different ways. We can impose priors on β_j and α using a hierarchical Bayes approach (e.g., Swaminathan & Gifford, 1982, 1985; Kim, Cohen, Baker, Subkoviak, & Leonard, 1994). If it is preferred that the priors not be too influential, uninformative priors could be imposed. Alternatively, it may also be useful to include external information in the form of fairly informative prior distributions. According to Spiegelhalter, Best, et al. (1996), it is important to avoid causal use of standard improper priors in MCMC modeling, since these may result in improper posterior distributions. Following Spiegelhalter, Thomas, et al. (1996), the uninformative prior distributions were chosen for the LSAT6 analyses to make comparisons with other estimation methods. The prior of β_j was $N(0, 100^2)$ and the prior of α was $N(0, 100^2)$ with the range restriction, $\alpha > 0$, to yield only positive values of the Gibbs sampler for the slope parameter. The prior distribution for α can be seen as a half normal distribution or the singly truncated normal distribution (Johnson, Kotz, & Balakrishnan, 1994). These prior distributions were similar to uninformative uniform distributions defined on the entire real line for β_j and on the positive real number line for α. An example input file for BUGS is given in Appendix.

Starting Values

The choice of starting values (e.g., $\omega^{(0)}$) is not generally that critical as the Gibbs sampler should be run long enough to be sufficiently updated from its initial states. It is useful, however, to perform a number of runs using different starting values to verify that the final results are not sensitive to the choice of starting values (Gelman, 1996). Raftery (1996) indicated that extreme starting values could lead to a very long burn-in or stabilization process.

To check the sensitivity of the starting values, three separate runs were performed using the LSAT6 data with three sets of starting values for β_j, $j = 1(1)5$, and α. The three sets of starting values are summarized in Table 2. The first run started at values considered plausible in the light of the usual range of item parameters. The second run and the third run represented substantial deviations in initial values. In particular, the second run was

intended to represent a situation in which there was a possibility that items were difficult, and the third run represented an opposite assumption.

Insert Table 2 about here

Each of the three runs consisted of 3,000 iterations. The results for β_1 are presented in Figure 3. The computer program CODA (Best, Cowles, & Vines, 1997) was used to obtain these graphs. The plots in Figure 3 contain the graphical summaries of the Gibbs sampler for β_1. The left plot shows the trace of the sampled values of β_1 for the three runs. In the legend, '0' indicates the initial values for β_j and α were 0 and 1, respectively; '5' indicates the initial values were 5 and 5, respectively; and, '-5' indicates initial values were -5 and .01, respectively. Results for all three runs show that the β_1 generated by the Gibbs sampler quickly settled down regardless of the starting values. The right graph shows the kernel density plot of the three pooled runs of 9,000 values for β_1. The variability among the β_1 values generated by the Gibbs sampler seems not to be too great. The sampled values seem to be concentrated around -2.5. The kernel density plot looks like a normal distribution.

Insert Figure 3 about here

The results for other item parameters were very similar to those from β_1. Overall, the starting values do not appear to affect the final results for the LSAT6 data. Useful starting values for the Rasch model can be found in Molenaar (1995), Gustafsson (1977), and Wright and Stone (1979). Also methods by Baker (1987), Jensema (1976), and Urry (1974) can be used to obtain starting values. Use of good starting values, such as from the above methods, can avoid the time delay required by a lengthy burn in. Our experience with these starting values indicates $\beta_j = 0$ and $\alpha = 1$ will work sufficiently well for applications under the 1PL. In subsequent analyses, therefore, the values, $\beta_j = 0$ and $\alpha = 1$, were used as starting values for LSAT6.

Output Monitoring

A critical issue for MCMC methods is how to determine when one can safely stop sampling and use the results to estimate characteristics of the distributions of the parameters of interest. In this regard, the values for the unknown quantities generated by the Gibbs

sampler can be graphically and statistically summarized to check mixing and convergence. The method proposed by Gelman and Rubin (1992) is one of the most popular for monitoring Gibbs sampling. Cowles and Carlin (1996) presented a comparative review of convergence diagnostics for the MCMC algorithms.

We illustrate, here, the use of Gelman and Rubin (1992) statistics on two 3,000 iteration runs. Details of the Gelman and Rubin method are also given in Gelman (1996). Each 3,000 iteration run required about 50 minutes on a Pentium 90 megahertz computer. Monitoring was done using the suite of S-functions called CODA (Best et al., 1997). Gelman-Rubin statistics (i.e., shrink factors) are plotted on Figure 4 for $\beta_1, \ldots, \beta_5$, and α, respectively. For all parameters, the medians were stabilized after about 1,000 iterations.

Insert Figure 4 about here

For each parameter, the Gelman-Rubin statistics estimate the reduction in the pooled estimate of variance if the runs were continued indefinitely. The Gelman-Rubin statistics can be calculated sequentially as the runs proceed. The Gelman-Rubin statistics should be near 1 in order to be reasonably assured that convergence has occurred. Table 3 contains the Gelman-Rubin statistics for LSAT6. The median for β_1, for example, was 1.00 and the 97.5 percentage point was 1.01. The median for α was 1.00 and the 97.5 percentage point was 1.02. These values were very close to 1 indicating that reasonable convergence was realized for all parameters. It is important to notice that the results in Table 3 and the plots in Figure 4 suggest the first 1,000 iterations of each run be discarded and the remaining samples be pooled. We used 1,000 iterations as burn-in and the subsequent 2,000 iterations for the estimation purpose.

Insert Table 3 about here

Item Parameter Estimates

The last step of Gibbs sampling is to obtain summary statistics for the quantities of interest. The posterior mean of the Gibbs sampler can be obtained for each item parameter. The posterior interval as well as the posterior standard deviation can also be obtained for each item parameter from the results of Gibbs sampling. In order to compare item parameter

estimates of the LSAT6 items, data were first analyzed via the computer program BUGS (Spiegelhalter et al., 1997) for Gibbs sampling using uninformative prior distributions for item parameters; $\beta_j \sim N(0, 100^2)$ and $\alpha \sim N(0, 100^2)$ with $\alpha > 0$. The starting values of the Gibbs sampler were $\beta_j = 0$ and $\alpha = 1$. There were the first 1,000 burn-in iterations. The subsequent 2,000 iterations were used to obtain posterior means and intervals of the item parameters. All of these were, of course, based on the results of the previous analyses presented earlier in this section. The trace lines of the sampled values and the kernel density plots for LSAT6 item parameters b_j, $j = 1(1)5$, and α are presented in Figure 5. All of the kernel density plots seem to follow the normal distributions.

Insert Figure 5 about here

Table 4 contains the Rasch item parameter estimates (i.e., posterior means for Gibbs sampling) and the 95% posterior intervals for the LSAT6 items. Table 4 also contains the item parameter estimates of the LSAT6 items from the methods of CML, MML, and JML using the computer programs PML (Molenaar, 1990), BILOG (Mislevy & Bock, 1990), and BIGSCALE (Wright, Linacre, & Schultz, 1989), respectively. All default options were used in running the programs. Note that the item parameter estimates from BILOG under MML were initially expressed in terms of the posterior ability metric. The item parameter estimates were transformed onto the usual Rasch model metric (i.e., the metric of either CML or JML with the restriction, $\sum_j b_j = 0$) in order to make the comparison possible.

All in all the item parameter estimates are the same. We also obtained correlations and root mean squared differences between sets of estimates for comparison purposes (see Table 5). The differences occurred mostly in the second or third decimal places. Considering the sizes of the confidence and posterior intervals of the estimates, there seem to be no practical differences in using the item parameter estimates for applications. In terms of confidence intervals, both MML and CML yielded relatively wider intervals than either Gibbs sampling or JML did.

Insert Tables 4 and 5 about here

Ability Parameter Estimates

The Rasch ability estimates and the posterior intervals of the LSAT6 data are reported in Table 6. It is important to notice that Gibbs sampling might yield different posterior means for examinees who have the same response pattern. For example, there were three examinees with the response pattern $(0, 0, 0, 0, 1)$ for LSAT6. If we obtain the posterior means for the three examinees, the values will be different (but obviously very similar). In this sense, estimates of the ability parameter from the Gibbs sampling are not unique if we try to obtain them jointly with item parameters. The ability estimates and the posterior intervals reported in Table 5 are, in fact, the average values based on the same raw scores.

Insert Tables 6 and 7 about here

The ability estimates from the methods of CML, MML, and JML can also be found in Table 6. Under MML, ability parameters are estimated after obtaining item parameter estimates and assuming the estimates are the true values. Three estimation methods, maximum likelihood (ML), expected a posteriori (EAP), and maximum a posteriori (MAP), can be used to obtain ability parameter estimates. Since the EAP methods is default in BILOG and since under CML and JML the ability estimates are based on the method of maximum likelihood, both EAP and ML methods were employed under MML. Note that the ability estimates were expressed in the same metric of the item parameter estimates.

In Table 7 it can be noticed that ability estimates from Gibbs sampling and EAP are about the same as both were based on the normal prior (i.e., Bayes methods). CML, MML/ML, and JML yielded very similar ability estimates. Especially, ability estimates and the confidence intervals from CML and MML/ML seem to be more similar each other than those from JML. Clearly, the Bayes ability estimates from both Gibbs sampling and EAP were different from those based on the maximum likelihood methods.

Example 2

Preliminary Analyses

The second example is based on the Memory Test data from Thissen (1982) (see Table 8). The Memory Test data contained 40 examinees responses to the ten items. This example may represent a situation where a small number of examinees' responses to a smaller number

of items are to be analyzed under the Rasch model. Model parameters were estimated by Gibbs sampling using the computer program BUGS (Spiegelhalter et al., 1997) under the 1PL model with the same sets of the prior distributions used in the LSAT6 analyses. That is, $\theta_i \sim N(0,1)$, $\beta_j \sim N(0,100^2)$, and $\alpha \sim N(0,100^2)$ with $\alpha > 0$.

Insert Table 8 about here

To check the sensitivity of the starting values for the Memory Test data, three separate runs were performed with three sets of starting values as in Table 2 for $\beta_j, j = 1(1)10$, and α. The three sets of starting values reflected such situations as we have items matched with ability, we have difficult items, and we have easy items, respectively. Each of the three runs consisted of 3,000 iterations.

The results for β_1 are presented in Figure 6. The left plot shows the trace of the sampled values of β_1 from the three runs. Results for all three runs indicated that the β_1 generated by the Gibbs sampler quickly settled down without any visible dependency on the starting values. The right graph shows the kernel density plot of the three pooled runs of 9,000 values for β_1. Variability among the β_1 values generated by the Gibbs sampler was very large, and it might reflect the fact that only 40 examinees were used to estimate parameters. The sampled values were concentrated around -1.5. The distribution did not reveal any bimodality or trimodality. The kernel density seemed to be a normal distribution indicating all three runs yielded similar sets of generated values that equally represented the underlying parameter β_1.

Insert Figure 6 about here

The results for other item parameters were almost the same as those for β_1. Overall the starting values do not appear to affect the final results for the Memory Test. In subsequent analyses for the Memory Test, therefore, $\beta_j = 0$ and $\alpha = 1$ were used as starting values.

The Gelman and Rubin (1992) statistics on two separate 3,000 iteration runs with different random number seeds were used to check mixing and convergence. Gelman-Rubin statistics are plotted on Figure 7 for $\beta_1, \ldots, \beta_{10}$, and α, respectively. In general, the medians were stabilized after about 1,000 iterations for all parameters.

Insert Figure 7 and Table 9 about here

Table 9 contains the Gelman-Rubin statistics of the Memory test. The median for α was 1.00 and the 97.5 percentage point was 1.02. The medians and the 97.5 percentage points for all β_j were 1.00. Reasonable convergence was achieved for all parameters. Note that Figure 7 suggests the first 1,000 iterations of each run be removed and the remaining samples be pooled. The first 1,000 iterations were treated as burn-in and the subsequent 2,000 iterations were used for making inferences.

Item Parameter Estimates

In order to compare item parameter estimates of the Memory Test, data were analyzed via the computer program BUGS (Spiegelhalter et al., 1997) for Gibbs sampling using the uninformative prior distributions for item parameters. Again $\beta_j \sim N(0, 100^2)$ and $\alpha \sim N(0, 100^2)$ with $\alpha > 0$ were used as priors. The starting values for Gibbs sampling were $\beta_j = 0$ and $\alpha = 1$. The last 2,000 iterations were used to obtain posterior means and posterior intervals of the item parameters. The trace lines of the sampled values and the kernel density plots for the Memory Test items parameters are presented in Figure 8. All of the kernel density plots seemed to follow the normal distributions.

Insert Figure 8 and Tables 10 and 11 about here

Table 10 contains the Rasch item parameter estimates and the 95% confidence and posterior intervals for the Memory Test items. Item parameter estimates were expressed in terms of the usual Rasch model metric. The item parameter estimates were very similar (see Table 11). The difference occurred mostly in the second decimal places and, sometimes, in the first decimal places. It can be noticed that the sizes of the confidence and posterior intervals were very large. This might not be surprising because there were only 40 examinees in the data. No practical differences, however, may occur in using these item parameter estimates. In terms of confidence intervals, MML and CML yielded relatively wider intervals than did the other two methods.

Ability Parameter Estimates

The Rasch ability estimates and the confidence and posterior intervals of the Memory Test data are reported in Table 12. Note that most of ability estimates and the posterior intervals from Gibbs sampling reported in Table 12 were the average values based on the same raw scores. The ability estimates from Gibbs sampling and EAP were very similar (see Table 13). CML, MML/ML, and JML yielded very similar ability estimates. The ability estimates and the confidence intervals from CML and MML/ML seemed to be more similar each other than those from JML. The Bayes ability estimates from Gibbs sampling and EAP were different from those based on the maximum likelihood methods.

Insert Tables 12 and 13 about here

Example 3

Preliminary Analyses

The third example used data from Patz and Junker (1997). The data consisted of 3,000 examinees' responses to six short constructed-response items from the 1992 Trial State Assessment in Reading of the National Assessment of Educational Progress (NAEP). According to Patz and Junker (1997), the sample of 3,000 examinees could be considered as a representative random sample of the population of the fourth grade students in the United States. The data provided a situation where a relatively short test was calibrated using a large number of examinees. Item response patterns of the six NAEP items and the numbers of examinees for the respective response patterns are displayed in Table 15. All 64 possible patterns were observed. The calibration was performed using BUGS (Spiegelhalter et al., 1997) under the 1PL. Prior distributions employed in calibration were $\theta_i \sim N(0,1)$, $\beta_j \sim N(0,100^2)$, and $\alpha \sim N(0,100^2)$ with $\alpha > 0$. It was expected that the relatively large sample size of the data would yield item parameter estimates that were not sensitive to the prior specifications because of the dominant effect of the likelihood in the posterior.

Insert Table 14 about here

Before making comparisons of calibration results, the effect of the starting values on the final parameter estimates was investigated for the NAEP data using three sets of starting

values as in Table 2. Each of the three runs consisted of 3,000 iterations. Figure 9 illustrates the convergence results of β_1 based on the three calibration runs. Each of the 3,000 iterations yielded very similar results. Regardless of starting values, the trace lines from the left plot were stabilized after just a few iterations. The kernel density plot of the combined 9,000 sampled values of β_1 is also presented in Figure 9. The kernel density plot shows all three starting values yielded the same pattern of sampled values. The density plot seems to follow a normal distribution. The results from other item parameters were very similar to the results of β_1. Overall the starting values did not appear to affect the final results for the NAEP items. The starting values, $\beta_j = 0$ and $\alpha = 1$, were used in the analyses.

Insert Figure 9 about here

In order to check mixing and convergence, Gelman and Rubin (1992) statistics were obtained from the two separate 3,000 iteration runs. Gelman-Rubin statistics are plotted in Figure 10. The medians were stabilized after about 1,000 iterations for all parameters. Hence, the first 1,000 iterations were treated as burn-in and the subsequent 2,000 iterations were used for estimating.

Insert Figure 10 and Table 15 about here

The Gelman-Rubin statistics for the parameters of the six NAEP items are presented in Table 15. The median for α was 1.00 and the 97.5 percentage point was 1.01. The medians for all β_j were 1.00. Three β_j yielded the 97.5 percentage points of 1.00. Two β_j (i.e., β_3 and β_5) yielded the 97.5 percentage points of 1.01. Reasonable convergence was realized for all parameters.

Item Parameter Estimates

In order to compare item parameter estimates of the NAEP items, data were analyzed via the computer program BUGS (Spiegelhalter et al., 1997) for Gibbs sampling using uninformative prior distributions for item parameters. The priors were $\beta_j \sim N(0, 100^2)$ and $\alpha \sim N(0, 100^2)$ with $\alpha > 0$. The starting values for Gibbs sampling were $\beta_j = 0$ and $\alpha = 1$. The last 2,000 iterations were used to obtain posterior means and posterior intervals of the item parameters.

The trace lines of the sampled values and the kernel density plots for the six NAEP items are presented in Figure 11. All of the kernel density plots seem to follow normal distributions.

Insert Figure 11 and Tables 16 and 17 about here

Table 16 contains the Rasch item parameter estimates and the 95% confidence and posterior intervals for the NAEP items from Gibbs sampling, CML, MML, and JML. Item parameter estimates were expressed in terms of the usual Rasch model metric. All four methods yielded almost the same item parameter estimates (see also Table 17). Gibbs sampling and MML yielded an identical set of item parameter estimates. The differences among item parameter estimates across estimation methods occurred mostly in the second decimal places. Gibbs sampling yielded relatively shorter posterior intervals than the other methods. The confidence and posterior intervals of the estimates were very short reflecting the fact that a total of 3,000 examinees were used to calibrate items. MML and CML yielded relatively wider confidence intervals than did the other two methods.

Ability Parameter Estimates

The Rasch ability estimates and the confidence and posterior intervals of the NAEP data are reported in Table 18. Note that the ability estimates and the posterior intervals for Gibbs sampling reported in Table 18 are the average values based on the same raw scores. The ability estimates from Gibbs sampling and EAP were very similar (see Table 19). CML, MML/ML, and JML yielded also very similar ability estimates. Among the maximum likelihood methods, the results from CML and MML/ML were more similar each other than those from JML. JML yielded relatively wider confidence intervals. Gibbs sampling yielded wider posterior intervals than EAP did except for scores 0 and 6. The Bayes ability estimates from Gibbs sampling and EAP were obviously quite different from those obtained from the other three maximum likelihood methods.

Insert Tables 18 and 19 about here

Example 4

Preliminary Analyses

The last Example represented a typical data set for 1PL that contained item responses from 365 examinees for the 31-item English Usage Test. The 1PL model with the prior distributions, $\theta_i \sim N(0,1)$, $\beta_j \sim N(0,100^2)$, and $\alpha \sim N(0,100^2)$ with $\alpha > 0$, was used in Gibbs sampling to estimate item and ability parameters.

To check the sensitivity of the starting values for the Usage Test data, three separate runs were performed with three sets of starting values for $\beta_j, j = 1(1)31$, and α, that were used in the previous Examples (see Table 2). The first starting values reflected a plausible set in the light of the usual range of item parameters. The second set represented a situation we have difficult items. The third implied that we have easy items. Each of the three runs consisted of 3,000 iterations. The results for β_1 are presented in Figure 12. The left plot shows that trace lines of the sampled values of β_1 for the three runs. The values of β_1 generated by the Gibbs sampler quickly settled down without having any visible effects of the starting values. The right graph shows the kernel density plot of the three pooled runs of 9,000 values of β_1. The shape of the kernel density was that of a normal distribution indicating all three runs yielded vary comparable sets that equally represented the underlying parameter β_1.

Insert Figure 12 about here

As the starting values did not appear to affect the final results for the Usage Test, the starting values, $\beta_j = 0$ and $\alpha = 1$, were used in the analyses. In addition, based on the results from the earlier Examples, the first 1,000 iterations were treated as burn-in and the next 2,000 iterations were used to obtain the posterior means and the posterior intervals of the item and ability parameters for Gibbs sampling.

Item Parameter Estimates

Table 20 contains the Rasch item parameter estimates and the 95% confidence and posterior intervals for the Usage Test items. Item parameter estimates were expressed in terms of the usual Rasch model metric. All item parameter estimates were very similar (see also Table 21). The differences among estimates occurred mostly in the second decimal places. In terms of confidence intervals, MML and CML yielded relatively wider intervals than the other two methods did.

Insert Tables 20 and 21 about here

Ability Parameter Estimates

The Rasch ability estimates and the confidence and posterior intervals of the Usage Test are reported in Table 22. Note that the ability estimates and the posterior intervals of Gibbs sampling reported were the average values based on the same raw scores. The ability estimates from Gibbs sampling and EAP were very similar (see Table 23). CML, MML/ML, and JML yielded very similar ability estimates. Bayes ability estimates from Gibbs sampling and EAP were clearly different from those obtained from the maximum likelihood methods.

Insert Tables 22 and 23 about here

Discussion

Previous work with the MCMC method using Gibbs sampling suggests this method may provide a useful alternative method for estimation when small sample sizes and small numbers of items are used. Even though implementation of the Gibbs sampling method in IRT is available in several computer programs, the accuracy of the resulting estimates have not been thoroughly studied. More simulation results should be reported.

The main difference between the Gibbs sampling method and the other estimation methods lies in the way these methods obtain parameter estimates. The Gibbs sampling method uses the sample of parameter values to estimate the mean and variance of the posterior density of the parameter. Under CML and MML, the conditional likelihood function and the marginalized likelihood function are maximized to obtain modes of item parameters. Estimates of the ability parameters do not arise during the course of item parameter estimation under CML and MML. Instead, ability parameters are typically estimated after obtaining the item parameter estimates, assuming the obtained estimates are true values. For the Gibbs sampling method, ability parameters can be estimated jointly with item parameters, similar, in this sense, to JML or joint Bayesian. It is important to know that the ability parameters can also be estimated in Gibbs sampling after obtaining item parameter estimates as in CML or MML, assuming the estimates are true values.

In the above context, one other difference between Gibbs sampling and the other estimation methods is that persons with the same response pattern may produce different ability estimates under Gibbs sampling. Clearly, it is not acceptable. Note that this will occur in a usual case of Gibbs sampling where both item and ability parameters are obtained jointly. We may perform Gibbs sampling initially only to estimate item parameters. After obtaining item parameter estimates, ability parameters can be obtained using a maximum likelihood or Bayesian method. It will remove such an awkward situation where examinees with the same response pattern have different ability estimates.

The estimation of item and ability parameters using Gibbs sampling requires a considerable amount of computing time. This was particularly true for the computer program BUGS used in this study. For example, as noted earlier, one computer run for Gibbs sampling using the LSAT6 data took about 50 minutes, whereas each of the other three estimation methods, MML, CML, and JML, took definitely less than a minute. The computer programs for MML, CML, and JML used in this study are extremely efficient, of course, in comparison to BUGS. One alternative solution may be implementing the Gibbs sampling method using lower level computer languages (e.g., FORTRAN or C++). The iterative nature of Gibbs sampling, however, may prohibit us from seeing a noticeable reduction of computing time.

The Gibbs sampling and general MCMC methods are likely to be more useful for situations where complicated models are employed. For example, Gibbs sampling can be applicable to the estimation of item and ability parameters in the hierarchical Bayes approach (Mislevy, 1986; Swaminathan & Gifford, 1982, 1985, 1986). In this study the priors were imposed directly on the parameters. Accuracy of the Gibbs sampling method with different kinds of priors, perhaps more informative in a Bayesian sense, should be investigated. This kind of research may be particularly valuable for small samples and short tests.

One of the possible advantages of using Gibbs sampling or general MCMC methods, and something to consider in future research on these methods, is incorporation of uncertainly in item parameter estimates into estimation of ability parameters (e.g. Patz & Junker, 1997; Tsutakawa & Johnson, 1990). The data sets used in the four Examples did not clearly exhibit any pronounced effects of errors in item parameter estimates on the ability estimates. This type of investigation can be performed in the context of simulation (e.g., Hulin, Lissak, & Drasgow, 1982). Additional simulation studies may reveal whether such incorporation is, in fact, valuable in the context of the Rasch model.

In this paper, The Rasch model was used without addressing the problem of model selection and criticism (e.g., the choice of the linking function, model fit). The model criticism for Gibbs sampling seems to be an important topic to investigate in future research. Also the evaluation of the Gibbs sampling method to other IRT models, for example, other logistic or probit models for binary items, the partial credit model, the graded response model, and the linear logistic test model, may provide guidelines for using the method under IRT.

Finally, it should be noted that the computer programs BUGS (Spiegelhalter et al., 1997) and CODA (Best et al., 1997) as well as the accompanying manuals are freely available over the Web. The uniform resource locator (URL) of the Medical Research Council Biostatistics Unit at the University of Cambridge is:

`http://www.mrc-bsu.cam.ac.uk/bugs/`

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Table 1

LSAT6 Data of Bock and Lieberman (1970) with 32 Response Patterns

Index	Item Pattern					Observed Freuqency
	1	2	3	4	5	
1	0	0	0	0	0	3
2	0	0	0	0	1	6
3	0	0	0	1	0	2
4	0	0	0	1	1	11
5	0	0	1	0	0	1
6	0	0	1	0	1	1
7	0	0	1	1	0	3
8	0	0	1	1	1	4
9	0	1	0	0	0	1
10	0	1	0	0	1	8
11	0	1	0	1	0	0
12	0	1	0	1	1	16
13	0	1	1	0	0	0
14	0	1	1	0	1	3
15	0	1	1	1	0	2
16	0	1	1	1	1	15
17	1	0	0	0	0	10
18	1	0	0	0	1	29
19	1	0	0	1	0	14
20	1	0	0	1	1	81
21	1	0	1	0	0	3
22	1	0	1	0	1	28
23	1	0	1	1	0	15
24	1	0	1	1	1	80
25	1	1	0	0	0	16
26	1	1	0	0	1	56
27	1	1	0	1	0	21
28	1	1	0	1	1	173
29	1	1	1	0	0	11
30	1	1	1	0	1	61
31	1	1	1	1	0	28
32	1	1	1	1	1	298

Table 2

Starting Values for Item Parameters in the Three Runs of the Gibbs Sampler

Parameter	Run		
	First	Second	Third
β_j, $j = 1(1)5$	0	5	−5
α	1	5	.01

Table 3

Gelman-Rubin Statistics for the Parameters of the LSAT6 Items

Parameter	Shrink Factor	
	Estimate	97.5 Percentile
β_1	1.00	1.01
β_2	1.00	1.01
β_3	1.00	1.01
β_4	1.00	1.01
β_5	1.00	1.01
α	1.00	1.02

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Table 4

Estimated Item Parameters and 95% Confidence/Posterior Intervals of the LSAT6 Items from Gibbs Sampling, Conditional Maximum Likelihood (CML), Marginal Maxium Likelihood (MML), and Joint Maximum Likelihood (JML)

	Gibbs Sampling[a]		CML		MML[a]		JML	
Item	Difficulty	Conf. Interval	Difficulty	Conf. Interval	Difficulty	Conf. Interval	Difficulty	Conf. Interval
1	−1.26	(−1.47, −1.05)	−1.26	(−1.49, −1.02)	−1.26	(−1.51, −1.00)	−1.24	(−1.46, −1.02)
2	.48	(.34, .62)	.47	(.31, .64)	.48	(.32, .63)	.45	(.31, .59)
3	1.24	(1.11, 1.37)	1.24	(1.08, 1.40)	1.24	(1.09, 1.38)	1.30	(1.16, 1.44)
4	.17	(.02, .31)	.17	(.00, .34)	.17	(.00, .34)	.13	(−.01, .27)
5	−.63	(−.79, −.47)	−.62	(−.82, −.43)	−.62	(−.83, −.42)	−.64	(−.80, −.48)

[a]The restriction, $\Sigma_j b_j = 0$, has been applied.

Table 5

Correlations (Lower Triangle) and Root Mean Squared Differences (Upper Triangle) of the LSAT6 Item Parameter Estimates from Gibbs Sampling, Conditional Maximum Likelihood (CML), Marginal Maxium Likelihood (MML), and Joint Maximum Likelihood (JML)

Method	Gibbs Sampling	CML	MML	JML
Gibbs Sampling		.006	.004	.036
CML	1.000		.004	.036
MML	1.000	1.000		.037
JML	.999	.999	.999	

Table 6

Ability Estimates and 95% Confidence/Posterior Intervals of the LSAT6 Data from Gibbs Sampling, Conditional Maximum Likelihood (CML), Marginal Maxium Likelihood (MML) with Maximum Likelihood (ML) and Expected A Posteriori (EAP), and Joint Maximum Likelihood (JML)

					MML[a]					
	Gibbs Sampling[a]		CML		ML		EAP		JML	
Score	Ability	Conf. Interval	Ability	Conf. Interval	Ability	Conf. Interval	Ability	Conf. Interval	Ability	Conf. Interval
0	.02	(−1.15, 1.18)					.03	(−1.14, 1.21)		
1	.39	(−.79, 1.60)	−1.60	(−3.92, .71)	−1.55	(***, ***[b])	.40	(−.78, 1.58)	−1.72	(−4.08, .65)
2	.76	(−.42, 1.98)	−.47	(−2.41, 1.47)	−.47	(−2.41, 1.47)	.76	(−.43, 1.96)	−.52	(−2.54, 1.50)
3	1.15	(−.06, 2.39)	.48	(−1.45, 2.42)	.48	(−1.45, 2.41)	1.14	(−.07, 2.36)	.52	(−1.50, 2.54)
4	1.54	(.31, 2.81)	1.60	(−.71, 3.91)	1.60	(−.71, 3.91)	1.54	(.29, 2.78)	1.72	(−.65, 4.09)
5	1.96	(.69, 3.26)					1.95	(.67, 3.23)		

[a]The restriction, $\Sigma_j b_j = 0$, has been applied.
[b]Improper values were obtained.

Table 7

Correlations (Lower Triangle) and Root Mean Squared Differences (Upper Triangle) of the LSAT6 Ability Estimates from Gibbs Sampling, Conditional Maximum Likelihood (CML), Marginal Maxium Likelihood (MML) with Maximum Likelihood (ML) and Expected A Posteriori (EAP), and Joint Maximum Likelihood (JML)

			MML		
Method	Gibbs Sampling	CML	ML	EAP	JML
Gibbs Sampling		1.217	1.197	.008	1.277
CML	.999		.025	1.220	.091
MML/ML	.927	.941		1.199	.109
MML/EAP	1.000	.999	.926		1.280
JML	.999	1.000	.939	.999	

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Table 8

Ten-Item Memory Test Data from Thissen (1982) with 31 Response Patterns

Index	Item Pattern										Observed Freuqency
	1	2	3	4	5	6	7	8	9	10	
1	0	0	0	0	0	0	0	0	0	0	5
2	0	0	0	0	0	0	0	0	0	1	1
3	0	0	0	0	0	0	0	0	1	1	3
4	0	0	0	0	0	0	0	1	0	1	2
5	0	0	0	0	0	1	0	0	0	1	1
6	0	0	0	0	1	0	0	0	0	1	1
7	0	0	0	0	1	0	0	0	1	0	1
8	0	0	1	0	0	0	0	0	0	1	1
9	0	0	0	0	0	0	0	1	1	1	2
10	0	0	0	0	0	0	1	0	1	1	1
11	0	0	1	0	0	0	0	1	0	1	1
12	0	0	1	0	0	0	1	0	0	1	1
13	0	1	0	0	0	1	0	1	0	0	1
14	1	0	0	0	0	0	0	0	1	1	1
15	1	0	0	0	0	0	1	0	0	1	1
16	1	0	0	1	0	0	0	0	1	0	1
17	0	0	0	0	0	0	1	1	1	1	1
18	0	0	0	0	0	1	0	1	1	1	2
19	0	0	0	0	1	0	1	0	1	1	1
20	0	0	0	1	0	0	1	0	1	1	1
21	0	0	0	1	0	0	1	1	0	1	1
22	0	1	0	0	0	0	0	1	1	1	1
23	0	1	0	0	0	1	0	0	1	1	1
24	0	1	0	0	1	0	0	1	1	0	1
25	0	1	0	0	0	0	1	1	1	1	1
26	1	0	0	0	0	1	1	1	0	1	1
27	1	0	0	1	1	0	1	1	0	0	1
28	1	1	0	0	1	0	0	1	0	1	1
29	0	1	0	0	0	1	1	1	1	1	1
30	1	1	0	0	1	1	0	1	0	1	1
31	0	1	1	1	1	0	0	1	1	1	1

Table 9

Gelman-Rubin Statistics for the Parameters of the Memory Test Items

Parameter	Shrink Factor	
	Estimate	97.5 Percentile
β_1	1.00	1.00
β_2	1.00	1.00
β_3	1.00	1.00
β_4	1.00	1.00
β_5	1.00	1.00
β_6	1.00	1.00
β_7	1.00	1.00
β_8	1.00	1.00
β_9	1.00	1.00
β_{10}	1.00	1.00
α	1.00	1.02

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Table 10

Estimated Item Parameters and 95% Confidence/Posterior Intervals of the Memory Test Items from Gibbs Sampling, Conditional Maximum Likelihood (CML), Marginal Mazium Likelihood (MML), and Joint Mazimum Likelihood (JML)

	Gibbs Sampling[a]		CML		MML[a]		JML	
Item	Difficulty	Conf. Interval	Difficulty	Conf. Interval	Difficulty	Conf. Interval	Difficulty	Conf. Interval
1	.70	(−.08, 1.51)	.66	(−.19, 1.51)	.65	(−.37, 1.67)	.68	(−.10, 1.46)
2	.31	(−.40, 1.09)	.33	(−.46, 1.12)	.31	(−.67, 1.30)	.35	(−.38, 1.08)
3	1.44	(.51, 2.61)	1.33	(.30, 2.36)	1.34	(.08, 2.60)	1.34	(.36, 2.32)
4	1.12	(.21, 2.11)	1.07	(.12, 2.02)	1.07	(−.05, 2.20)	1.09	(.19, 1.99)
5	.50	(−.25, 1.33)	.49	(−.33, 1.31)	.47	(−.48, 1.43)	.51	(−.23, 1.25)
6	.49	(−.25, 1.28)	.49	(−.33, 1.31)	.47	(−.50, 1.45)	.51	(−.23, 1.25)
7	.01	(−.68, .73)	.05	(−.71, .80)	.02	(−.82, .85)	.06	(−.63, .75)
8	−1.01	(−1.68, −.34)	−.91	(−1.62, −.20)	−.96	(−1.75, −.16)	−.93	(−1.58, −.28)
9	−1.13	(−1.79, −.44)	−1.02	(−1.74, −.31)	−1.07	(−1.93, −.21)	−1.05	(−1.70, −.40)
10	−2.43	(−3.26, −1.69)	−2.49	(−3.38, −1.59)	−2.31	(−3.25, −1.36)	−2.58	(−3.46, −1.70)

[a]The restriction, $\Sigma_j b_j = 0$, has been applied.

Table 11

Correlations (Lower Triangle) and Root Mean Squared Differences (Upper Triangle) of the Memory Test Item Parameter Estimates from Gibbs Sampling, Conditional Mazimum Likelihood (CML), Marginal Mazium Likelihood (MML), and Joint Mazimum Likelihood (JML)

Method	Gibbs Sampling	CML	MML	JML
Gibbs Sampling		.066	.061	.072
CML	.999		.063	.034
MML	1.000	.999		.090
JML	.998	1.000	.998	

Table 12

Ability Estimates and 95% Confidence/Posterior Intervals of the Memory Test Data from Gibbs Sampling, Conditional Mazimum Likelihood (CML), Marginal Mazium Likelihood (MML) with Mazimum Likelihood (ML) and Expected A Posteriori (EAP), and Joint Mazimum Likelihood (JML)

					MML[a]					
	Gibbs Sampling[a]		CML		ML		EAP		JML	
Score	Ability	Conf. Interval	Ability	Conf. Interval	Ability	Conf. Interval	Ability	Conf. Interval	Ability	Conf. Interval
0	−2.24	(−3.63, −.89)					−2.05	(−3.23, −.88)		
1	−1.84	(−3.17, −.58)	−2.71	(−4.98, −.44)	−2.67	(−4.91, −.44)	−1.70	(−2.84, −.56)	−2.78	(−5.05, −.51)
2	−1.47	(−2.74, −.23)	−1.69	(−3.45, .06)	−1.69	(−3.42, .05)	−1.37	(−2.48, −.27)	−1.72	(−3.48, .04)
3	−1.11	(−2.33, .07)	−1.00	(−2.54, .54)	−1.00	(−2.53, .52)	−1.06	(−2.14, .02)	−.99	(−2.54, .56)
4	−.78	(−2.01, .39)	−.43	(−1.86, 1.00)	−.44	(−1.86, .98)	−.76	(−1.82, .30)	−.41	(−1.84, 1.02)
5	−.46	(−1.68, .71)	.08	(−1.30, 1.45)	.06	(−1.31, 1.44)	−.48	(−1.52, .57)	.11	(−1.26, 1.48)
6	−.15	(−1.31, .99)	.57	(−.81, 1.95)	.56	(−.83, 1.94)	−.20	(−1.23, .83)	.60	(−.77, 1.97)
7	.17	(−.99, 1.35)	1.09	(−.36, 2.53)	1.07	(−.38, 2.52)	.08	(−.95, 1.10)	1.12	(−.31, 2.55)

[a]The restriction, $\Sigma_j b_j = 0$, has been applied.

Table 13

Correlations (Lower Triangle) and Root Mean Squared Differences (Upper Triangle) of the Memory Test Ability Estimates from Gibbs Sampling, Conditional Mazimum Likelihood (CML), Marginal Mazium Likelihood (MML) with Mazimum Likelihood (ML) and Expected A Posteriori (EAP), and Joint Mazimum Likelihood (JML)

			MML		
Method	Gibbs Sampling	CML	ML	EAP	JML
Gibbs Sampling		.609	.592	.100	.642
CML	.995		.019	.672	.036
MML/ML	.995	1.000		.654	.054
MML/EAP	1.000	.994	.995		.223
JML	.994	1.000	1.000	.993	

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Table 14

NAEP Data in Patz and Junker (1997) with 64 Response Patterns

Index	Item Pattern						Observed Frequency	Index	Item Pattern						Observed Freuqency
	1	2	3	4	5	6			1	2	3	4	5	6	
1	0	0	0	0	0	0	145	33	1	0	0	0	0	0	13
2	0	0	0	0	0	1	44	34	1	0	0	0	0	1	8
3	0	0	0	0	1	0	49	35	1	0	0	0	1	0	7
4	0	0	0	0	1	1	16	36	1	0	0	0	1	1	5
5	0	0	0	1	0	0	13	37	1	0	0	1	0	0	4
6	0	0	0	1	0	1	2	38	1	0	0	1	0	1	1
7	0	0	0	1	1	0	17	39	1	0	0	1	1	0	3
8	0	0	0	1	1	1	6	40	1	0	0	1	1	1	5
9	0	0	1	0	0	0	141	41	1	0	1	0	0	0	22
10	0	0	1	0	0	1	49	42	1	0	1	0	0	1	9
11	0	0	1	0	1	0	79	43	1	0	1	0	1	0	20
12	0	0	1	0	1	1	45	44	1	0	1	0	1	1	16
13	0	0	1	1	0	0	22	45	1	0	1	1	0	0	3
14	0	0	1	1	0	1	14	46	1	0	1	1	0	1	1
15	0	0	1	1	1	0	21	47	1	0	1	1	1	0	10
16	0	0	1	1	1	1	18	48	1	0	1	1	1	1	11
17	0	1	0	0	0	0	157	49	1	1	0	0	0	0	34
18	0	1	0	0	0	1	47	50	1	1	0	0	0	1	16
19	0	1	0	0	1	0	104	51	1	1	0	0	1	0	36
20	0	1	0	0	1	1	65	52	1	1	0	0	1	1	33
21	0	1	0	1	0	0	37	53	1	1	0	1	0	0	6
22	0	1	0	1	0	1	28	54	1	1	0	1	0	1	3
23	0	1	0	1	1	0	32	55	1	1	0	1	1	0	20
24	0	1	0	1	1	1	40	56	1	1	0	1	1	1	30
25	0	1	1	0	0	0	265	57	1	1	1	0	0	0	40
26	0	1	1	0	0	1	106	58	1	1	1	0	0	1	33
27	0	1	1	0	1	0	202	59	1	1	1	0	1	0	60
28	0	1	1	0	1	1	177	60	1	1	1	0	1	1	98
29	0	1	1	1	0	0	64	61	1	1	1	1	0	0	19
30	0	1	1	1	0	1	46	62	1	1	1	1	0	1	26
31	0	1	1	1	1	0	107	63	1	1	1	1	1	0	50
32	0	1	1	1	1	1	93	64	1	1	1	1	1	1	107

Table 15

Gelman-Rubin Statistics for the Parameters of the NAEP Items

Parameter	Shrink Factor	
	Estimate	97.5 Percentile
β_1	1.00	1.00
β_2	1.00	1.00
β_3	1.00	1.01
β_4	1.00	1.00
β_5	1.00	1.01
β_6	1.00	1.00
α	1.00	1.01

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Table 16

Estimated Item Parameters and 95% Confidence/Posterior Intervals of the NAEP Items from Gibbs Sampling, Conditional Maximum Likelihood (CML), Marginal Mazium Likelihood (MML), and Joint Mazimum Likelihood (JML)

	Gibbs Sampling[a]		CML		MML[a]		JML	
Item	Difficulty	Conf. Interval	Difficulty	Conf. Interval	Difficulty	Conf. Interval	Difficulty	Conf. Interval
1	1.14	(1.07, 1.23)	1.15	(1.05, 1.24)	1.14	(1.05, 1.24)	1.13	(1.05, 1.24)
2	−1.27	(−1.34, −1.19)	−1.26	(−1.35, −1.17)	−1.27	(−1.36, −1.17)	−1.25	(−1.33, −1.17)
3	−.89	(−.97, −.82)	−.89	(−.98, −.81)	−.89	(−.98, −.81)	−.88	(−.96, −.81)
4	.93	(.85, 1.00)	.93	(.84, 1.02)	.93	(.84, 1.02)	.92	(.84, 1.00)
5	−.26	(−.33, −.19)	−.26	(−.35, −.18)	−.26	(−.34, −.17)	−.25	(−.33, −.17)
6	.35	(.28, .42)	.34	(.26, .43)	.35	(.26, .43)	.34	(.26, .42)

[a]The restriction, $\Sigma_j b_j = 0$, has been applied.

Table 17

Correlations (Lower Triangle) and Root Mean Squared Differences (Upper Triangle) of the NAEP Item Parameter Estimates from Gibbs Sampling, Conditional Mazimum Likelihood (CML), Marginal Mazium Likelihood (MML), and Joint Mazimum Likelihood (JML)

Method	Gibbs Sampling	CML	MML	JML
Gibbs Sampling		.007	.000	.012
CML	1.000		.007	.011
MML	1.000	1.000		.012
JML	1.000	1.000	1.000	

Table 18

Ability Estimates and 95% Confidence/Posterior Intervals of the NAEP Data from Gibbs Sampling, Conditional Mazimum Likelihood (CML), Marginal Mazium Likelihood (MML) with Mazimum Likelihood (ML) and Expected A Posteriori (EAP), and Joint Mazimum Likelihood (JML)

					MML[a]					
	Gibbs Sampling[a]		CML		ML		EAP		JML	
Score	Ability	Conf. Interval	Ability	Conf. Interval	Ability	Conf. Interval	Ability	Conf. Interval	Ability	Conf. Interval
0	−1.29	(−2.65, −.03)					−1.33	(−2.65, −.01)		
1	−.88	(−2.20, .40)	−1.86	(−4.12, .39)	−1.87	(−4.12, .39)	−.89	(−2.17, .39)	−1.96	(−4.25, .33)
2	−.49	(−1.77, .75)	−.82	(−2.66, 1.02)	−.82	(−2.66, 1.02)	−.47	(−1.73, .78)	−.87	(−2.77, 1.03)
3	−.06	(−1.44, 1.17)	.01	(−1.75, 1.76)	.01	(−1.75, 1.76)	−.07	(−1.31, 1.18)	.01	(−1.79, 1.81)
4	.35	(−.92, 1.64)	.83	(−1.01, 2.66)	.83	(−1.01, 2.66)	.34	(−.91, 1.59)	.88	(−1.00, 2.76)
5	.74	(−.56, 2.07)	1.86	(−.38, 4.10)	1.86	(−.38, 4.11)	.75	(−.52, 2.02)	1.96	(−.31, 4.23)
6	1.18	(−.07, 2.45)					1.18	(−.12, 2.48)		

[a]The restriction, $\Sigma_j b_j = 0$, has been applied.

Table 19

Correlations (Lower Triangle) and Root Mean Squared Differences (Upper Triangle) of the NAEP Ability Estimates from Gibbs Sampling, Conditional Mazimum Likelihood (CML), Marginal Mazium Likelihood (MML) with Mazimum Likelihood (ML) and Expected A Posteriori (EAP), and Joint Mazimum Likelihood (JML)

			MML		
Method	Gibbs Sampling	CML	ML	EAP	JML
Gibbs Sampling		.715	.718	.018	.785
CML	.998		.004	.713	.071
MML/ML	.998	1.000		.716	.068
MML/EAP	1.000	.999	.999		.783
JML	.998	1.000	1.000	.999	

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Table 20

Estimated Item Parameters and 95% Confidence/Posterior Intervals of the English Usage Items from Gibbs Sampling, Conditional Maximum Likelihood (CML), Marginal Maximum Likelihood (MML), and Joint Maximum Likelihood (JML)

	Gibbs Sampling[a]		CML		MML[a]		JML	
Item	Difficulty	Conf. Interval	Difficulty	Conf. Interval	Difficulty	Conf. Interval	Difficulty	Conf. Interval
1	−2.67	(−3.12, −2.24)	−2.68	(−3.13, −2.23)	−2.65	(−3.13, −2.17)	−2.68	(−3.13, −2.23)
2	1.15	(.92, 1.38)	1.14	(.90, 1.38)	1.14	(.90, 1.38)	1.14	(.90, 1.38)
3	1.20	(.96, 1.44)	1.20	(.96, 1.43)	1.20	(.97, 1.43)	1.20	(.96, 1.44)
4	1.92	(1.67, 2.17)	1.91	(1.64, 2.18)	1.91	(1.66, 2.16)	1.92	(1.67, 2.17)
5	−.97	(−1.23, −.73)	−.96	(−1.23, −.70)	−.97	(−1.24, −.70)	−.97	(−1.22, −.72)
6	−.62	(−.87, −.37)	−.61	(−.86, −.36)	−.62	(−.87, −.36)	−.62	(−.86, −.38)
7	−.65	(−.91, −.41)	−.64	(−.89, −.39)	−.65	(−.90, −.39)	−.65	(−.89, −.41)
8	.40	(.17, .62)	.39	(.16, .62)	.39	(.17, .62)	.39	(.17, .61)
9	.82	(.59, 1.04)	.81	(.58, 1.04)	.81	(.59, 1.04)	.81	(.59, 1.03)
10	.53	(.30, .76)	.52	(.29, .75)	.52	(.29, .76)	.52	(.30, .74)
11	.17	(−.05, .39)	.17	(−.06, .40)	.17	(−.06, .39)	.17	(−.05, .39)
12	−.37	(−.61, −.13)	−.37	(−.61, −.13)	−.37	(−.61, −.14)	−.37	(−.61, −.13)
13	1.00	(.77, 1.22)	.99	(.75, 1.22)	.99	(.77, 1.21)	.99	(.75, 1.23)
14	−1.55	(−1.85, −1.27)	−1.55	(−1.86, −1.24)	−1.55	(−1.85, −1.25)	−1.55	(−1.84, −1.26)
15	1.29	(1.05, 1.52)	1.28	(1.04, 1.52)	1.29	(1.03, 1.54)	1.29	(1.05, 1.53)
16	1.09	(.88, 1.30)	1.08	(.85, 1.32)	1.09	(.86, 1.31)	1.09	(.85, 1.33)
17	−.87	(−1.12, −.62)	−.86	(−1.12, −.60)	−.86	(−1.13, −.60)	−.86	(−1.11, −.61)
18	−.59	(−.83, −.34)	−.58	(−.83, −.33)	−.58	(−.82, −.35)	−.58	(−.82, −.34)
19	−.64	(−.89, −.40)	−.64	(−.89, −.39)	−.65	(−.90, −.39)	−.65	(−.89, −.41)
20	−1.28	(−1.57, −1.00)	−1.28	(−1.56, −.99)	−1.28	(−1.58, −.98)	−1.28	(−1.55, −1.01)
21	.17	(−.05, .39)	.17	(−.06, .40)	.17	(−.06, .40)	.17	(−.05, .39)
22	.77	(.55, .98)	.76	(.53, .99)	.76	(.53, .99)	.76	(.54, .98)
23	.54	(.32, .76)	.55	(.32, .78)	.55	(.33, .77)	.55	(.33, .77)
24	.76	(.53, .98)	.76	(.53, .99)	.76	(.52, 1.00)	.76	(.54, .98)
25	−1.92	(−2.26, −1.59)	−1.91	(−2.25, −1.56)	−1.90	(−2.26, −1.55)	−1.91	(−2.24, −1.58)
26	−.54	(−.78, −.31)	−.53	(−.78, −.29)	−.54	(−.79, −.28)	−.54	(−.78, −.30)
27	.06	(−.18, .29)	.06	(−.17, .29)	.06	(−.17, .29)	.06	(−.16, .28)
28	−.46	(−.70, −.23)	−.46	(−.70, −.22)	−.46	(−.70, −.22)	−.46	(−.70, −.22)
29	2.17	(1.88, 2.44)	2.16	(1.88, 2.44)	2.15	(1.87, 2.43)	2.17	(1.90, 2.44)
30	−.08	(−.32, .14)	−.07	(−.31, .16)	−.08	(−.31, .16)	−.08	(−.32, .16)
31	−.80	(−1.06, −.54)	−.79	(−1.05, −.53)	−.79	(−1.06, −.53)	−.79	(−1.04, −.54)

[a]The restriction, $\Sigma_j b_j = 0$, has been applied.

Table 21

Correlations (Lower Triangle) and Root Mean Squared Differences (Upper Triangle) of the English Usage Item Parameter Estimates from Gibbs Sampling, Conditional Maximum Likelihood (CML), Marginal Maximum Likelihood (MML), and Joint Maximum Likelihood (JML)

Method	Gibbs Sampling	CML	MML	JML
Gibbs Sampling		.008	.009	.006
CML	1.000		.008	.005
MML	1.000	1.000		.007
JML	1.000	1.000	1.000	

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Table 22

Ability Estimates and 95% Confidence/Posterior Intervals of the English Usage Test Data from Gibbs Sampling, Conditional Maximum Likelihood (CML), Marginal Maximum Likelihood (MML) with Maximum Likelihood (ML) and Expected A Posteriori (EAP), and Joint Maximum Likelihood (JML)

					MML[a]					
	Gibbs Sampling[a]		CML		ML		EAP		JML	
Score	Ability	Conf. Interval	Ability	Conf. Interval	Ability	Conf. Interval	Ability	Conf. Interval	Ability	Conf. Interval
0	−2.69	(−3.90, −1.68)					−2.65	(−3.64, −1.66)		
2	−2.17	(−3.16, −1.29)	−3.17	(−4.68, −1.67)	−3.17	(−4.67, −1.67)	−2.14	(−3.11, −1.16)	−3.20	(−4.71, −1.69)
4	−1.74	(−2.66, −.93)	−2.31	(−3.44, −1.18)	−2.31	(−3.44, −1.18)	−1.73	(−2.53, −.92)	−2.33	(−3.47, −1.19)
5	−1.55	(−2.45, −.71)	−2.00	(−3.04, −.97)	−2.00	(−3.04, −.97)	−1.55	(−2.39, −.72)	−2.03	(−3.07, −.99)
6	−1.37	(−2.27, −.56)	−1.74	(−2.72, −.77)	−1.74	(−2.71, −.77)	−1.36	(−2.26, −.46)	−1.76	(−2.74, −.78)
7	−1.20	(−2.04, −.39)	−1.51	(−2.43, −.58)	−1.51	(−2.43, −.58)	−1.15	(−2.02, −.28)	−1.52	(−2.44, −.60)
8	−1.03	(−1.87, −.26)	−1.29	(−2.18, −.41)	−1.30	(−2.18, −.41)	−.98	(−1.73, −.23)	−1.31	(−2.19, −.43)
9	−.86	(−1.63, −.10)	−1.10	(−1.96, −.24)	−1.10	(−1.96, −.24)	−.85	(−1.50, −.21)	−1.11	(−1.97, −.25)
10	−.71	(−1.48, .07)	−.91	(−1.75, −.07)	−.91	(−1.75, −.07)	−.74	(−1.41, −.07)	−.92	(−1.76, −.08)
11	−.58	(−1.37, .19)	−.73	(−1.55, .09)	−.73	(−1.55, .09)	−.61	(−1.40, .18)	−.74	(−1.56, .08)
12	−.42	(−1.20, .32)	−.56	(−1.37, .25)	−.56	(−1.37, .25)	−.42	(−1.30, .45)	−.57	(−1.37, .23)
13	−.27	(−1.01, .48)	−.39	(−1.19, .40)	−.39	(−1.19, .40)	−.23	(−1.06, .60)	−.40	(−1.20, .40)
14	−.13	(−.88, .58)	−.23	(−1.02, .56)	−.23	(−1.02, .56)	−.08	(−.77, .62)	−.23	(−1.01, .55)
15	.02	(−.73, .77)	−.07	(−.85, .72)	−.07	(−.86, .72)	.03	(−.56, .62)	−.07	(−.85, .71)
16	.15	(−.57, .89)	.09	(−.69, .88)	.09	(−.69, .88)	.12	(−.49, .73)	.09	(−.69, .87)
17	.30	(−.44, 1.09)	.26	(−.53, 1.04)	.26	(−.53, 1.04)	.24	(−.51, .98)	.26	(−.52, 1.04)
18	.45	(−.34, 1.21)	.42	(−.38, 1.21)	.42	(−.38, 1.21)	.41	(−.45, 1.27)	.42	(−.38, 1.22)
19	.57	(−.20, 1.33)	.58	(−.22, 1.39)	.58	(−.22, 1.39)	.61	(−.25, 1.46)	.59	(−.21, 1.39)
20	.74	(−.01, 1.47)	.75	(−.06, 1.57)	.75	(−.06, 1.57)	.78	(.03, 1.52)	.76	(−.06, 1.58)
21	.89	(.10, 1.64)	.93	(.10, 1.76)	.93	(.10, 1.76)	.90	(.25, 1.55)	.94	(.10, 1.78)
22	1.04	(.28, 1.83)	1.11	(.26, 1.97)	1.11	(.26, 1.97)	1.01	(.33, 1.69)	1.13	(.27, 1.99)
23	1.20	(.42, 1.97)	1.31	(.43, 2.19)	1.31	(.43, 2.19)	1.16	(.35, 1.96)	1.32	(.44, 2.20)
24	1.37	(.56, 2.20)	1.52	(.60, 2.43)	1.52	(.60, 2.43)	1.35	(.45, 2.25)	1.53	(.61, 2.45)
25	1.56	(.71, 2.42)	1.75	(.79, 2.71)	1.75	(.79, 2.71)	1.56	(.68, 2.44)	1.76	(.80, 2.72)
26	1.72	(.88, 2.61)	2.00	(.98, 3.02)	2.00	(.98, 3.02)	1.75	(.92, 2.57)	2.02	(1.00, 3.04)
27	1.95	(1.05, 2.90)	2.30	(1.18, 3.41)	2.30	(1.18, 3.41)	1.92	(1.08, 2.77)	2.32	(1.20, 3.44)
28	2.16	(1.26, 3.15)	2.66	(1.41, 3.90)	2.66	(1.41, 3.90)	2.13	(1.19, 3.08)	2.68	(1.43, 3.93)
29	2.40	(1.45, 3.45)	3.13	(1.65, 4.61)	3.13	(1.65, 4.61)	2.39	(1.37, 3.40)	3.16	(1.67, 4.65)
30	2.68	(1.66, 3.83)	3.90	(1.87, 5.93)	3.90	(1.87, 5.92)	2.66	(1.62, 3.71)	3.92	(1.90, 5.94)
31	2.98	(1.88, 4.18)					2.97	(1.85, 4.08)		

[a]The restriction, $\Sigma_j b_j = 0$, has been applied.

Table 23

Correlations (Lower Triangle) and Root Mean Squared Differences (Upper Triangle) of the English Usage Ability Estimates from Gibbs Sampling, Conditional Maximum Likelihood (CML), Marginal Maximum Likelihood (MML) with Maximum Likelihood (ML) and Expected A Posteriori (EAP), and Joint Maximum Likelihood (JML)

			MML		
Method	Gibbs Sampling	CML	ML	EAP	JML
Gibbs Sampling		.403	.404	.031	.417
CML	.996		.000	.415	.016
MML/ML	.996	1.000		.416	.015
MML/EAP	1.000	.996	.996		.429
JML	.996	1.000	1.000	.996	

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Figure Captions

Figure 1. The Gibbs sampling algorithm.

Figure 2. A directed acyclic graph for LSAT6 data.

Figure 3. Convergence with starting values for LSAT6 item 1.

Figure 4. Gelman and Rubin shrink factors for LSAT6 items.

Figure 5. Trace lines of the sampled values and kernel density plots for LSAT6 items.

Figure 6. Convergence with starting values for Memory Test item 1.

Figure 7. Gelman and Rubin shrink factors for Memory Test items.

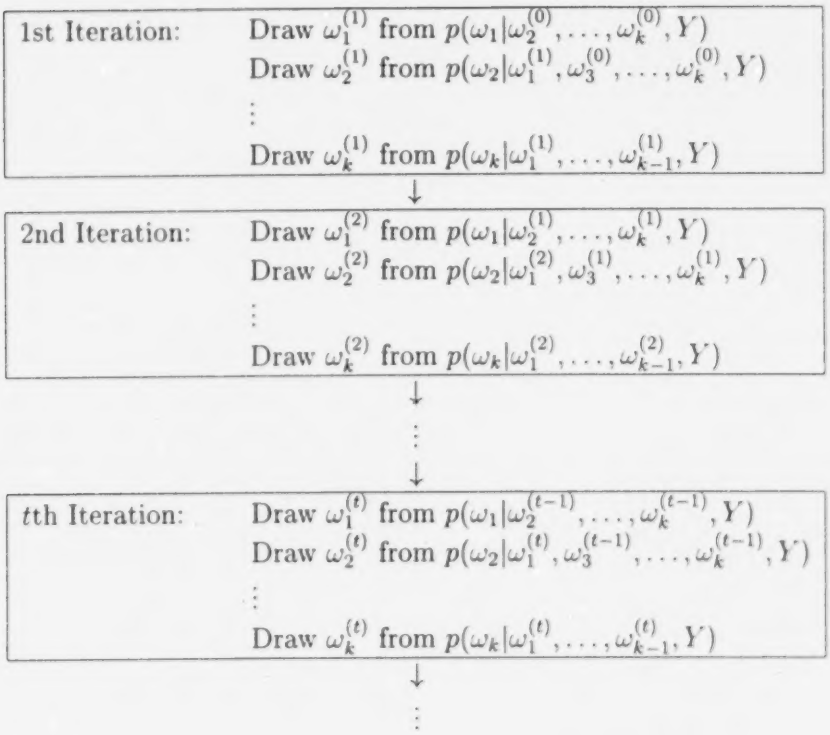
Figure 8. Trace lines of the sampled values and kernel density plots for Memory Test items.

Figure 9. Convergence with starting values for NAEP item 1.

Figure 10. Gelman and Rubin shrink factors for NAEP items.

Figure 11. Trace lines of the sampled values and kernel density plots for NAEP items.

Figure 12. Convergence with starting values for Usage item 1.



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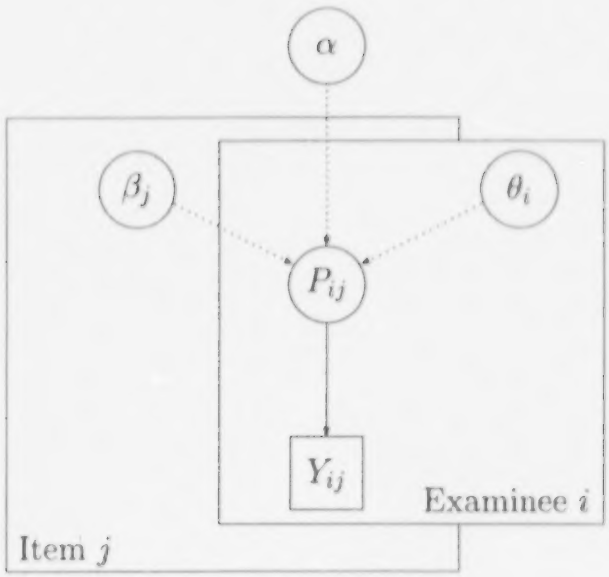
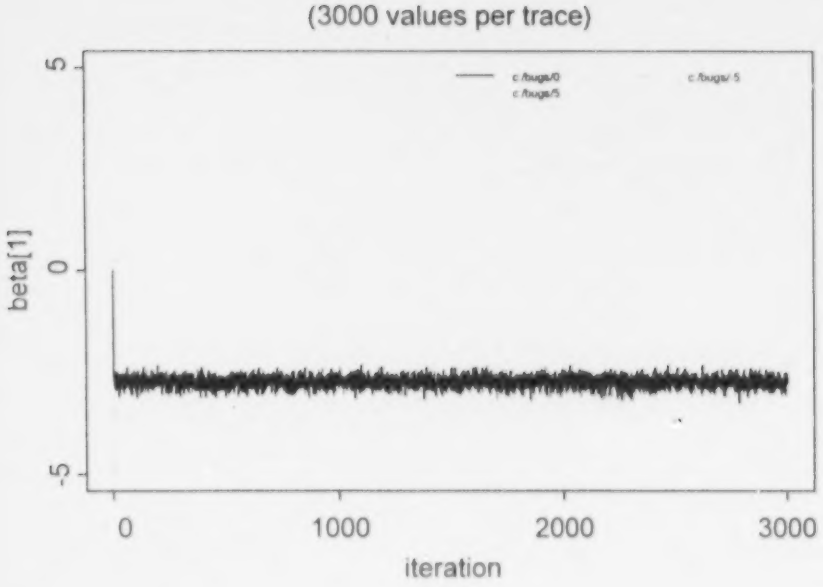
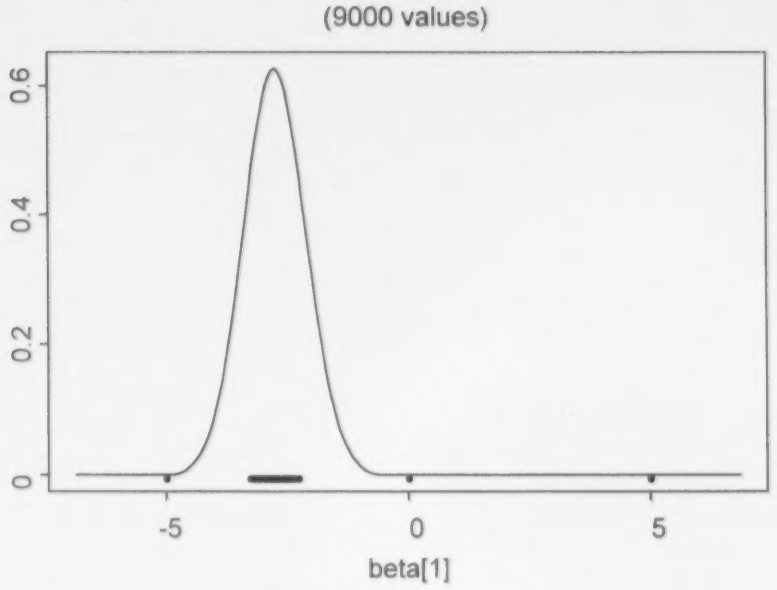

α
β_j
θ_i
P_{ij}
Y_{ij}
Examinee i
Item j

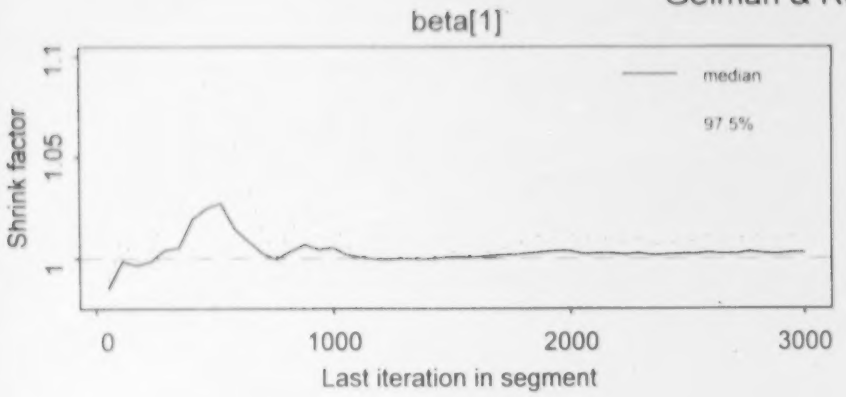
Convergence with Starting Values for LSAT6 Item-1

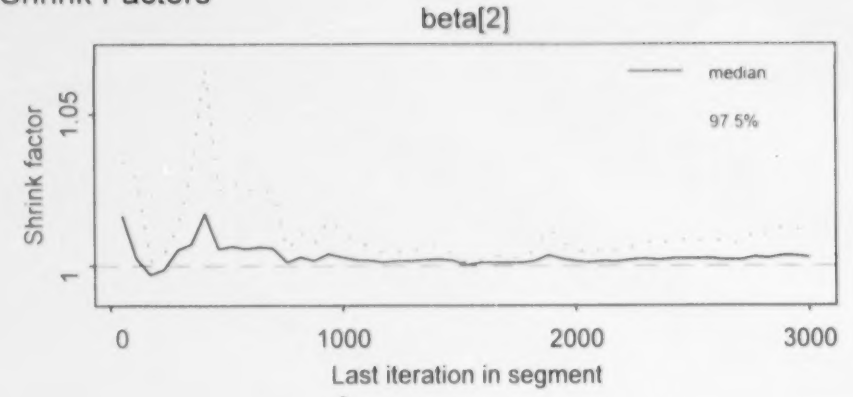


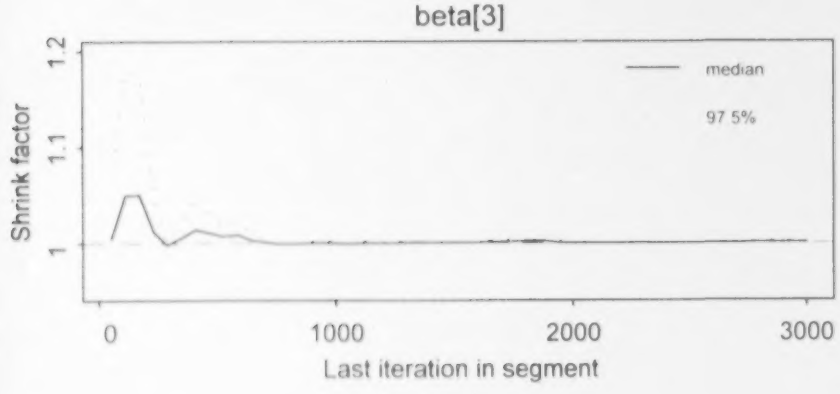


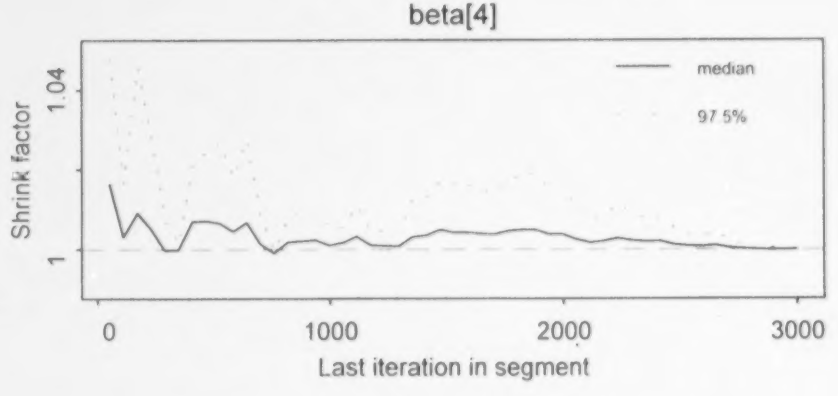
LSAT6 Items

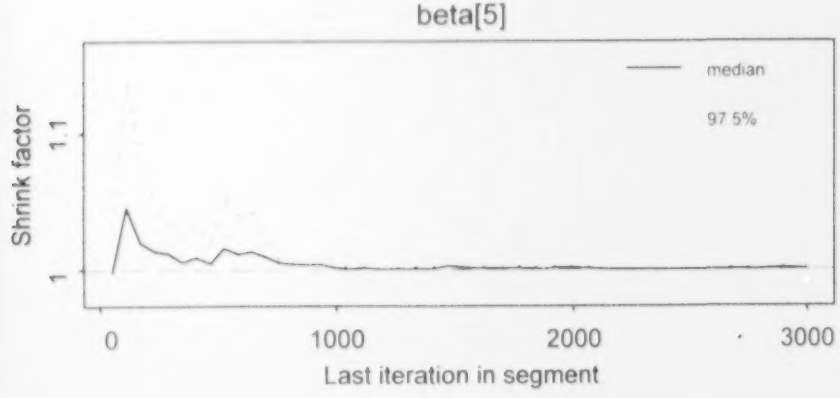
Gelman & Rubin Shrink Factors

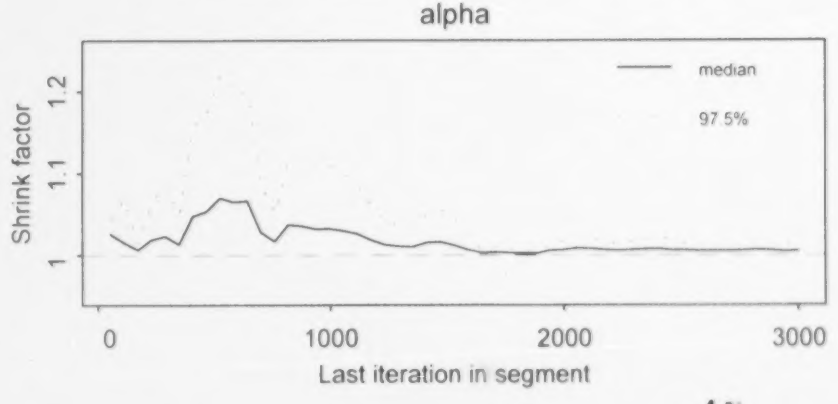




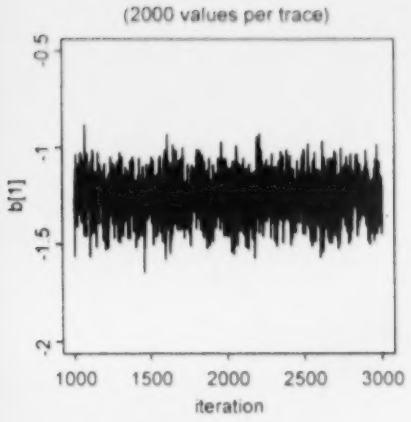








LSAT6 Items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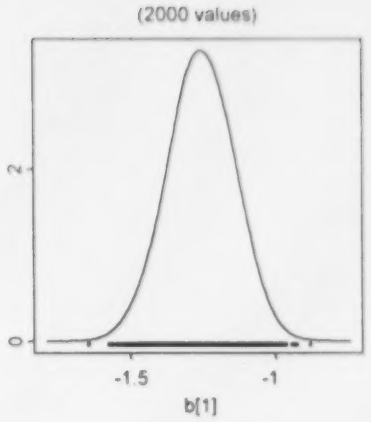
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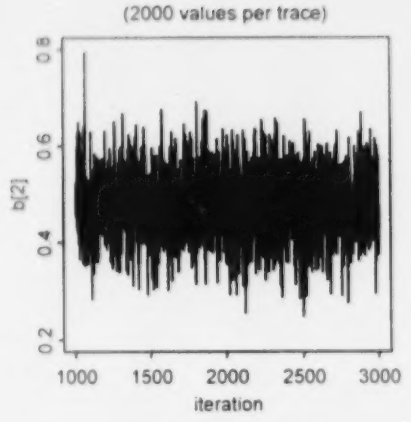
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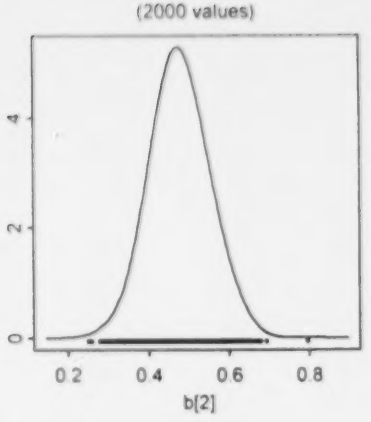
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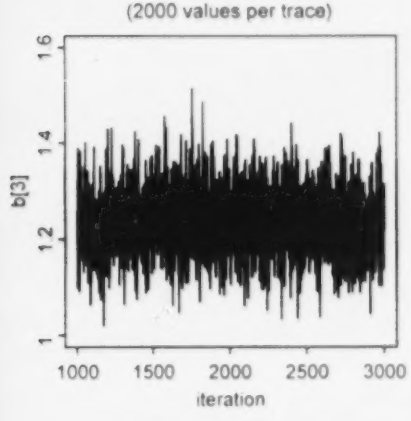
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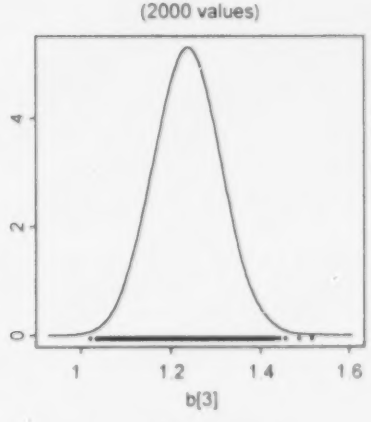
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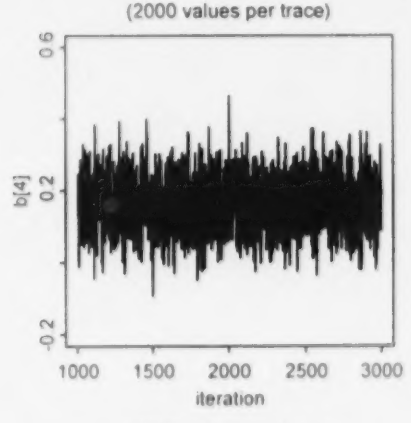
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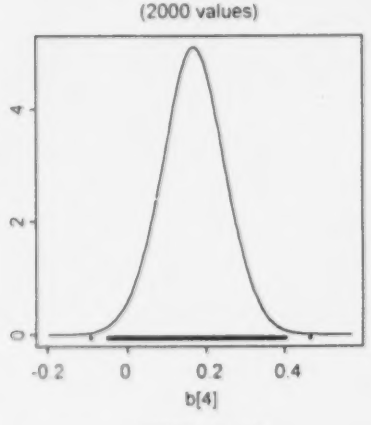
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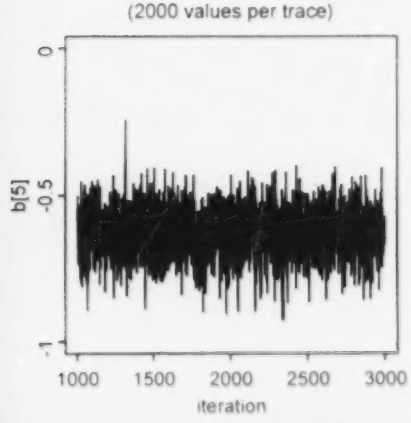
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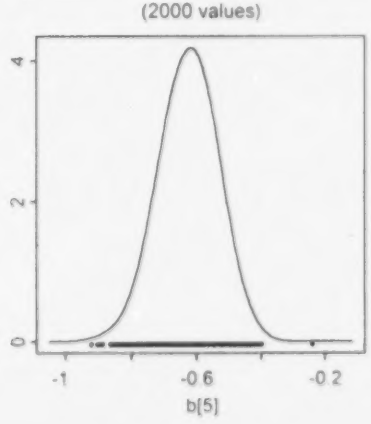
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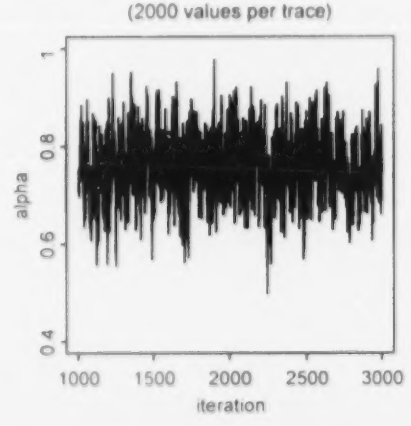
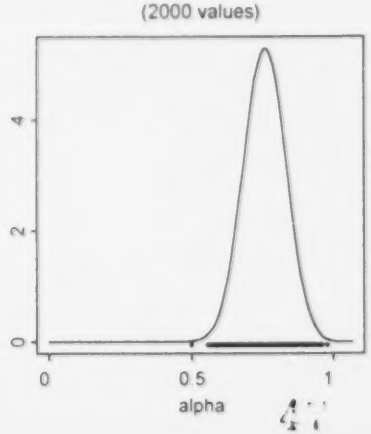
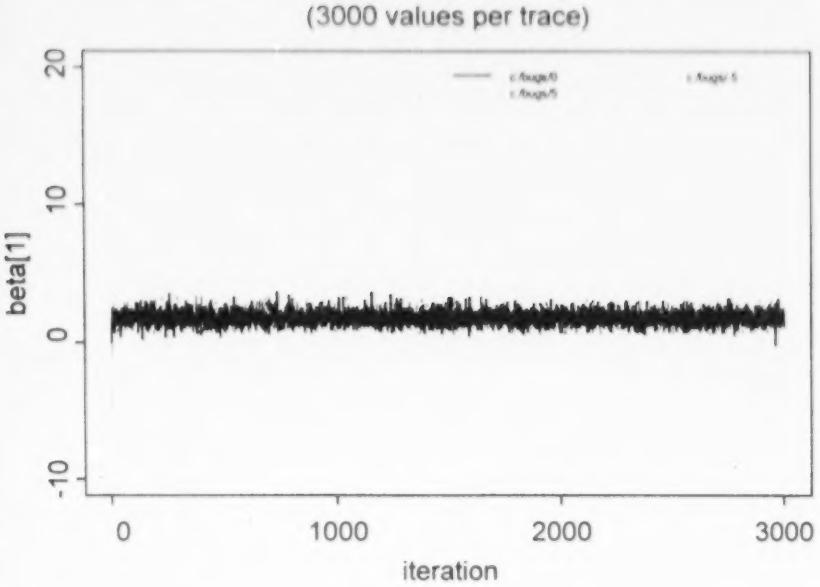
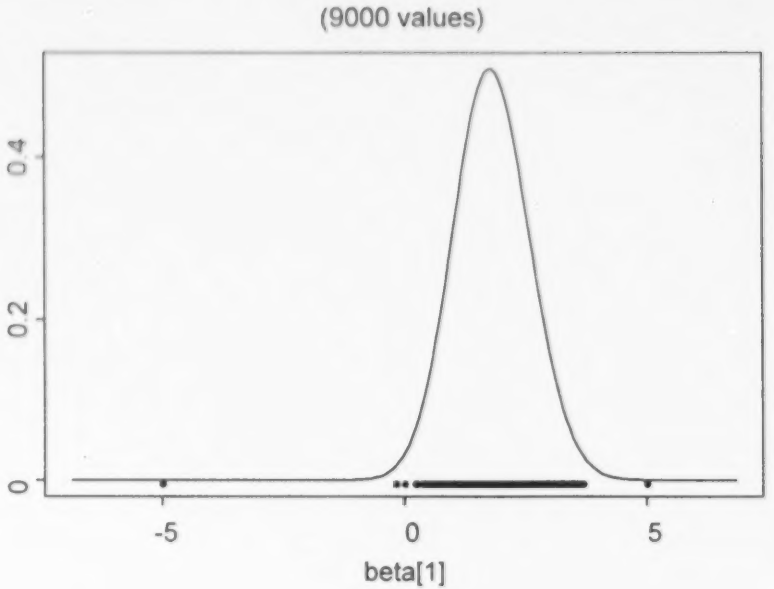

(2000 values per trace)
alpha
iteration


(2000 values)
alpha

Convergence with Starting Values for Memory Test Item-1





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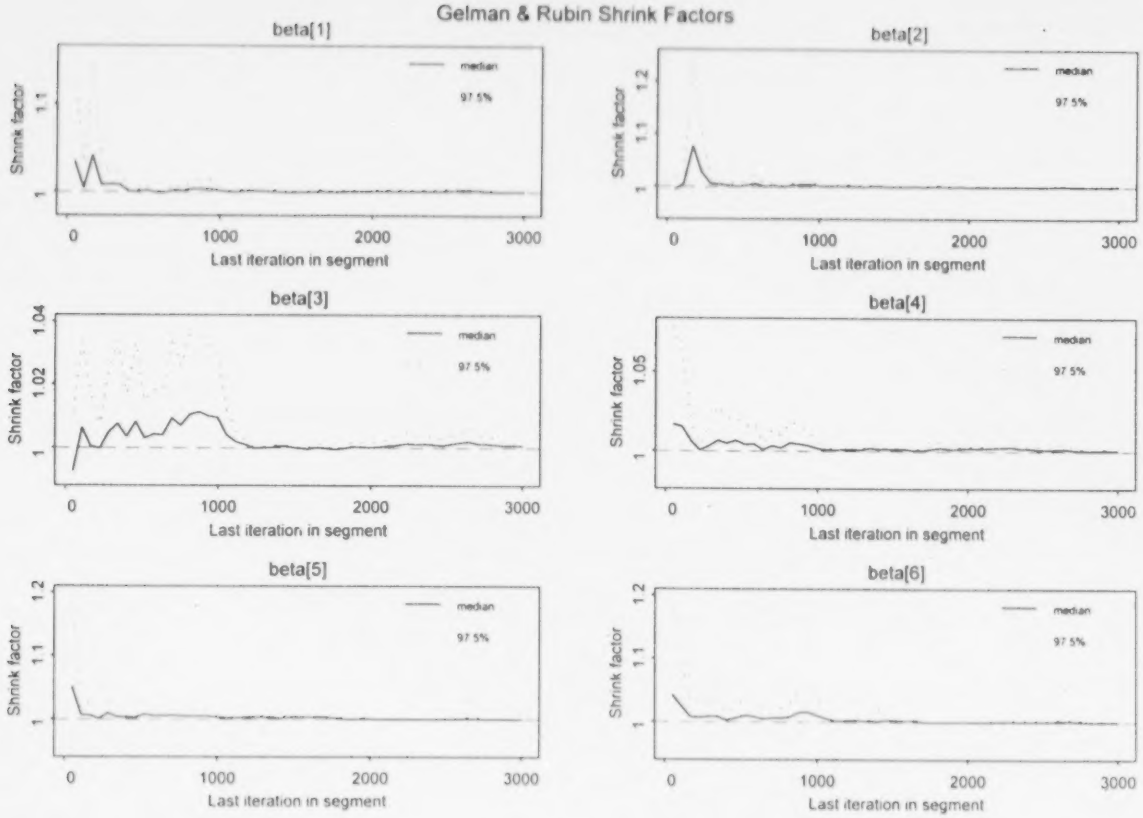
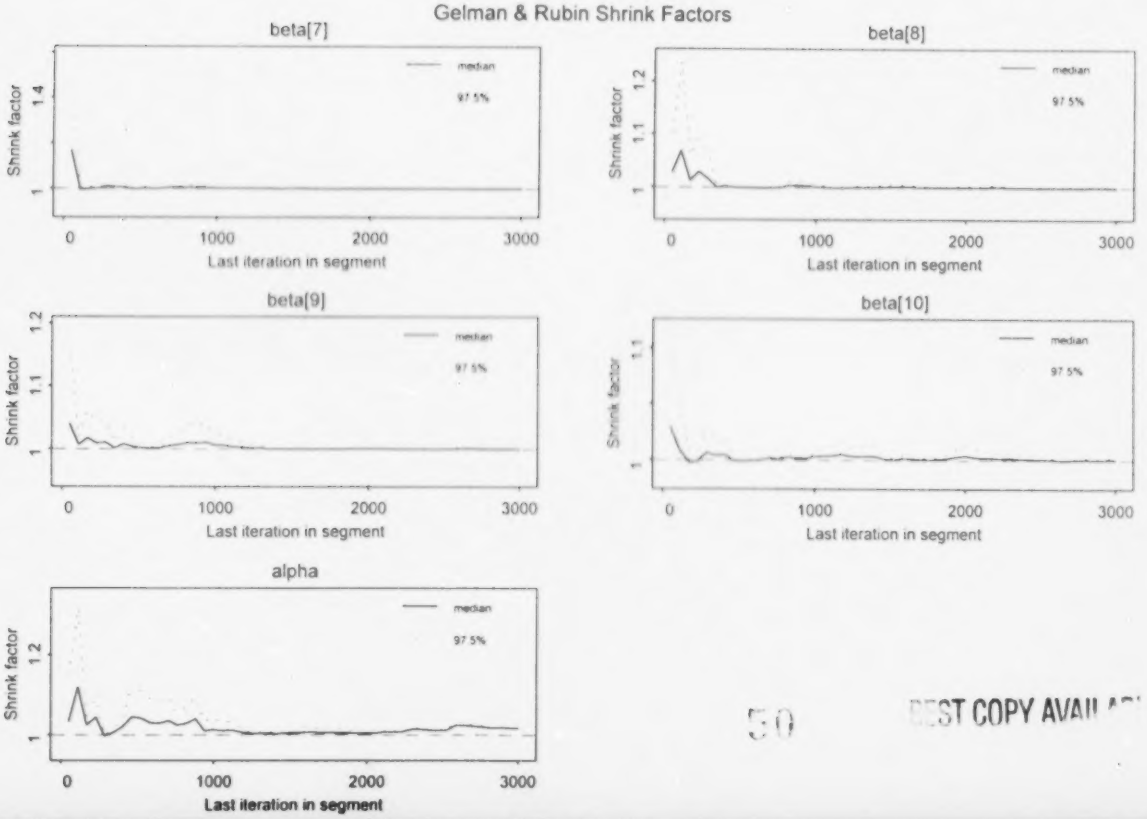
Memory Test Items

Gelman & Rubin Shrink Factors



Memory Test Items

Gelman & Rubin Shrink Factors



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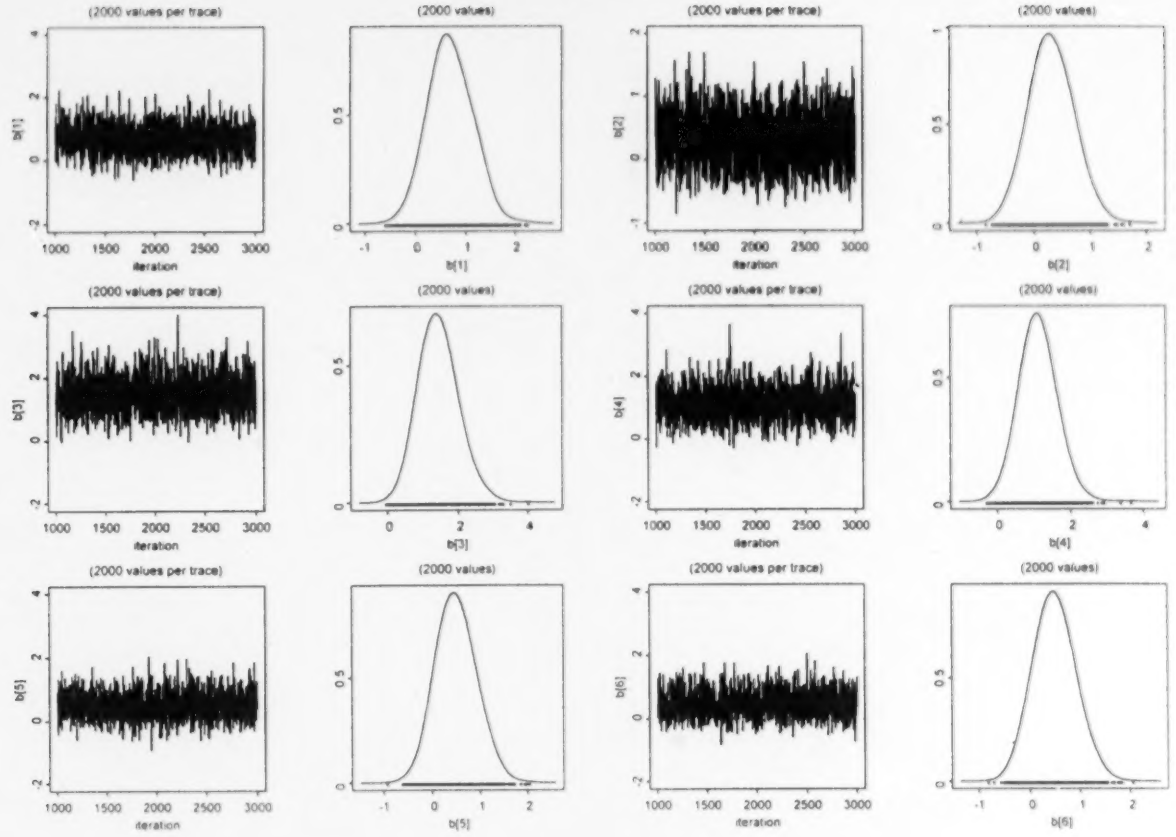
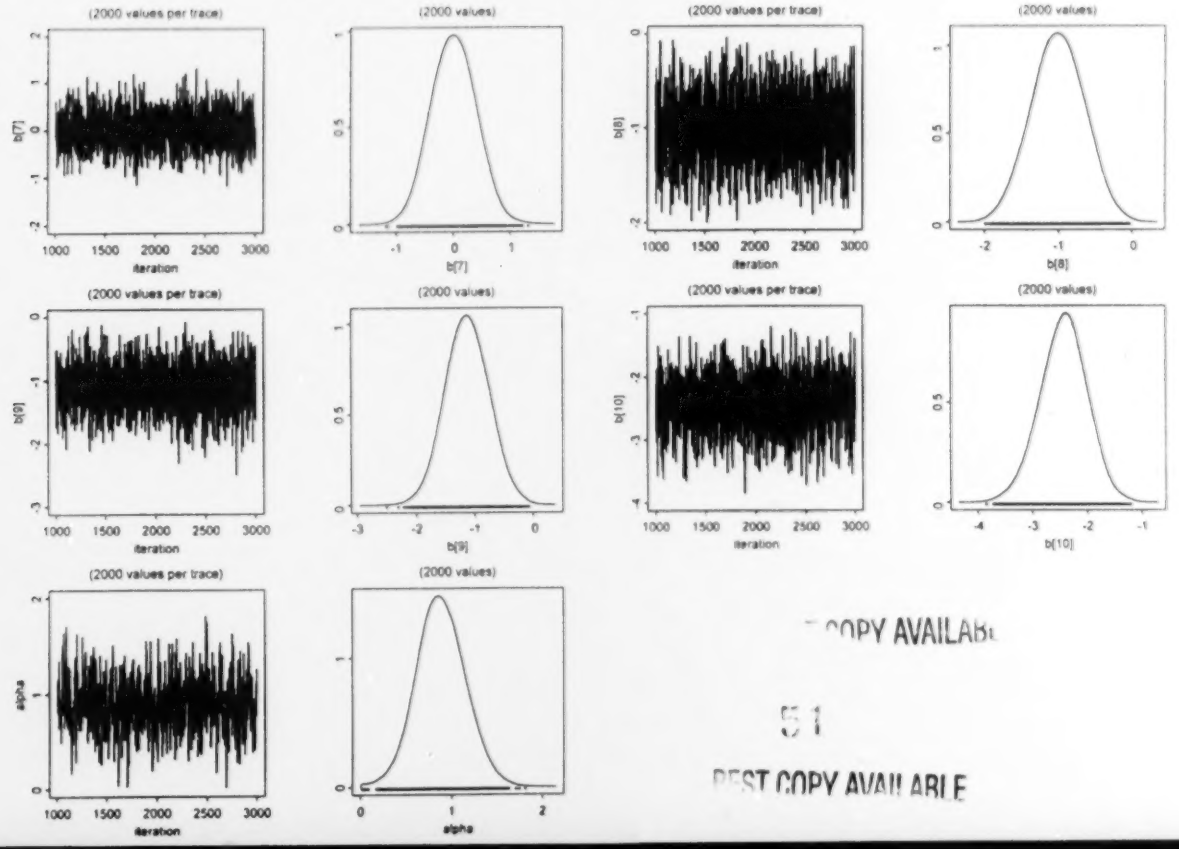
Memory Test Items



Memory Test Items



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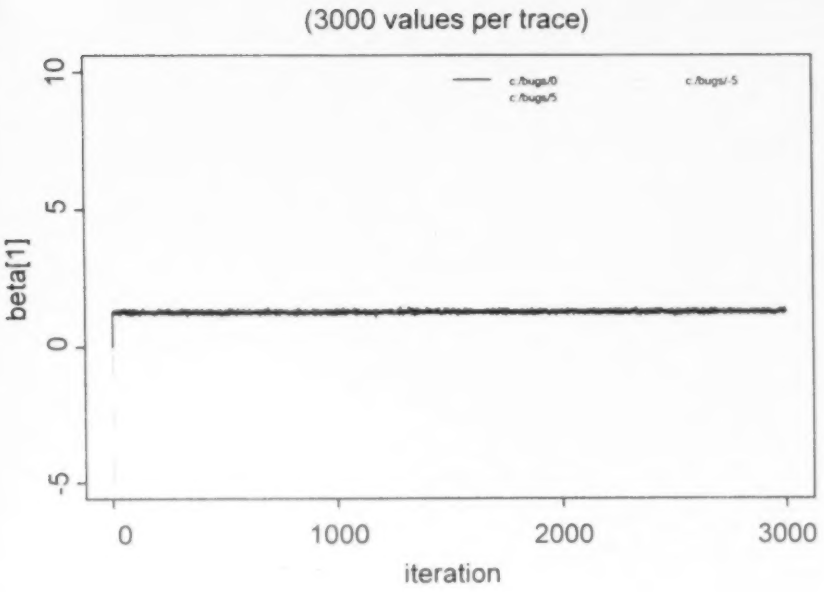
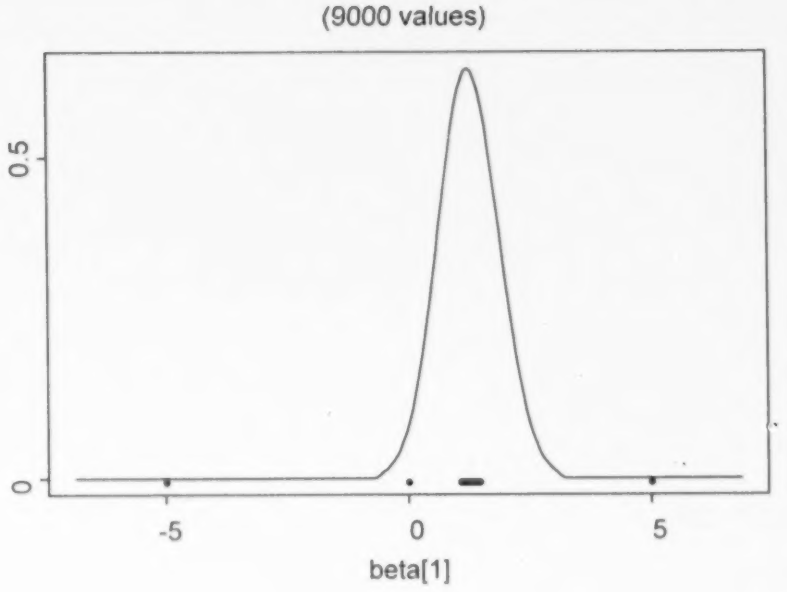
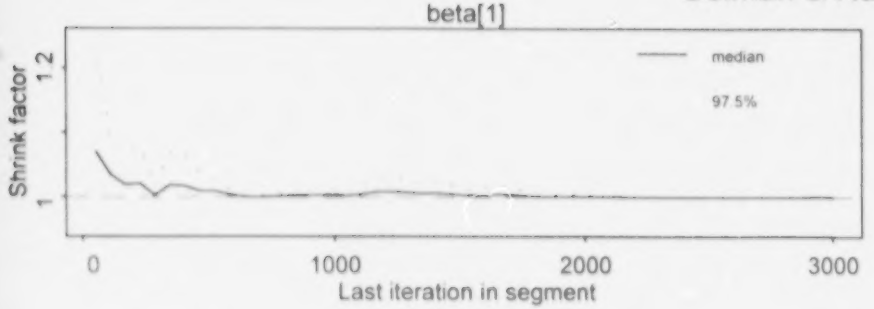
Convergence with Starting Values for NAEP Item-1

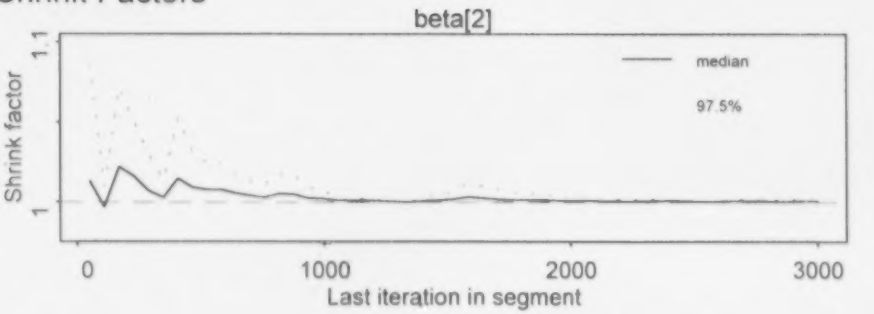


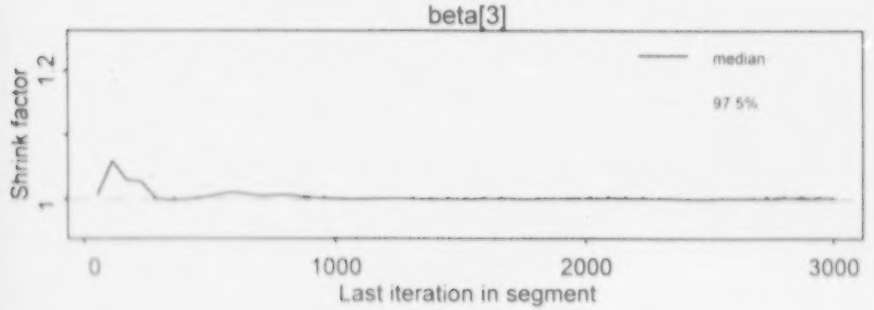


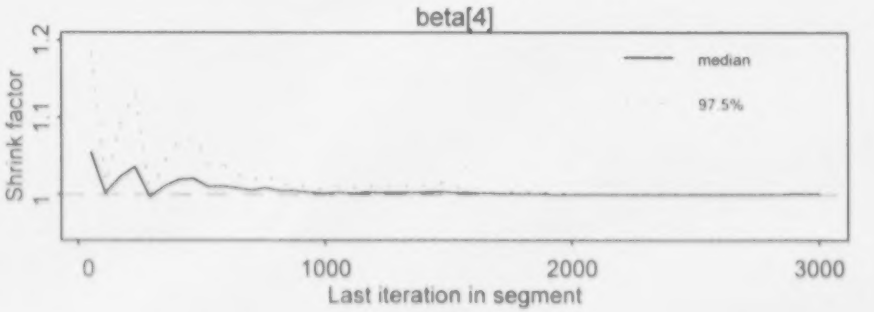
NAEP Items

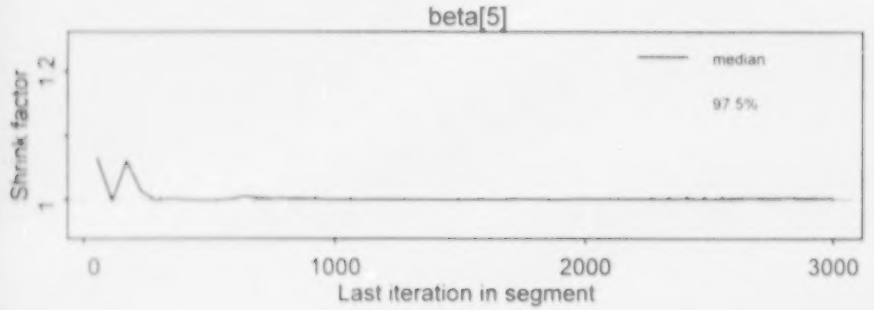
Gelman & Rubin Shrink Factors

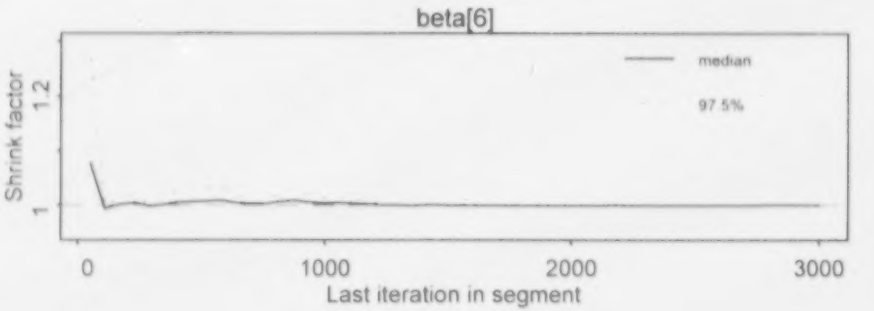


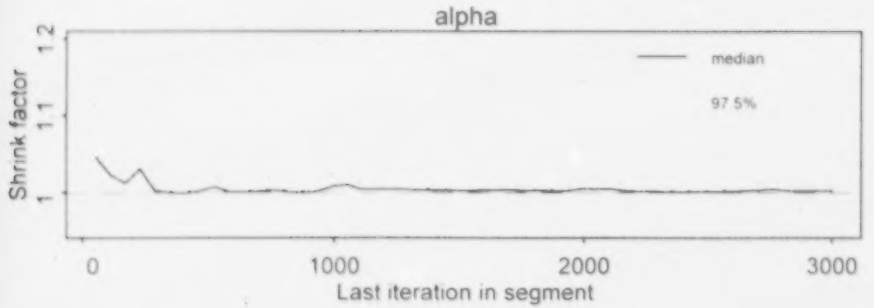




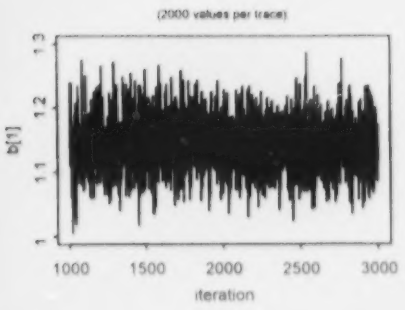








NAEP Items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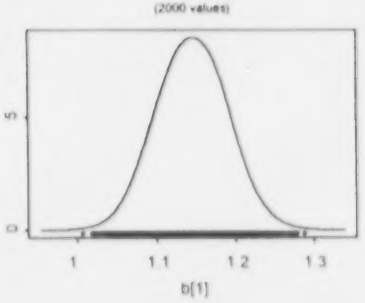
b[1]
iteration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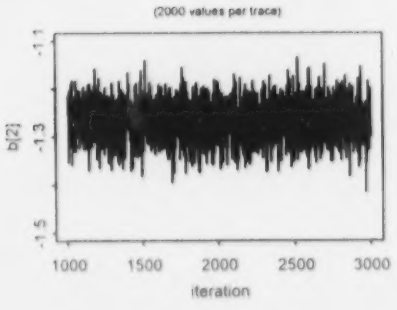
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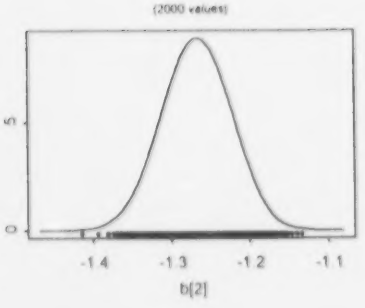
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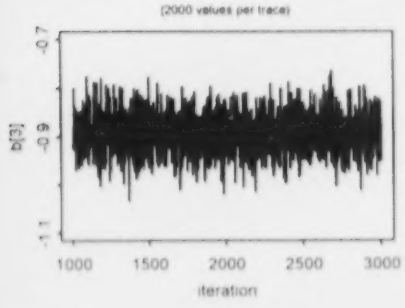
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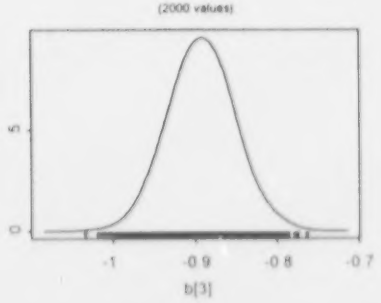
b[3]
iteration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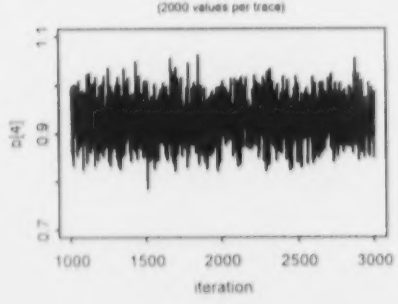
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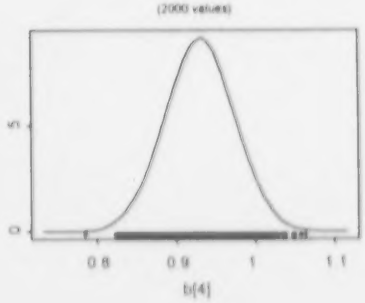
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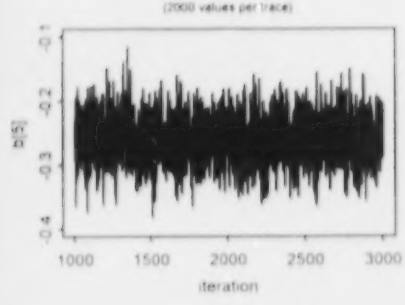
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b[5]
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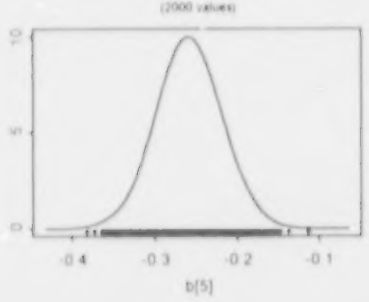
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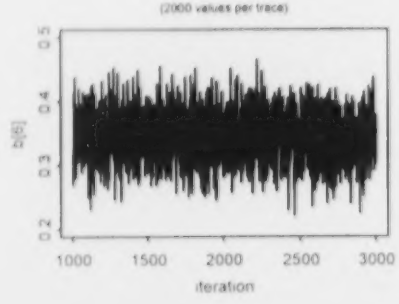
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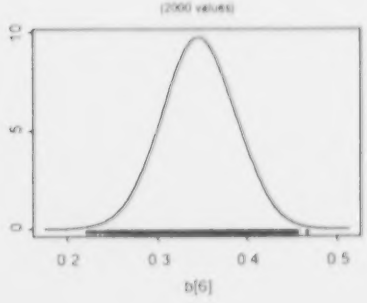
b[6]
iteration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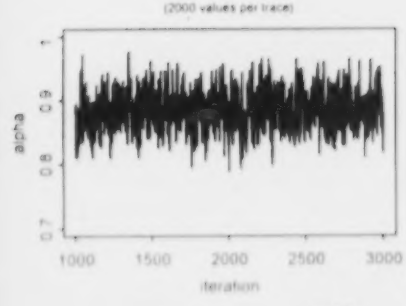
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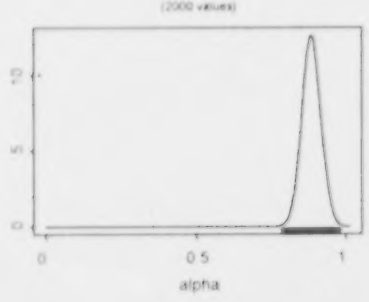
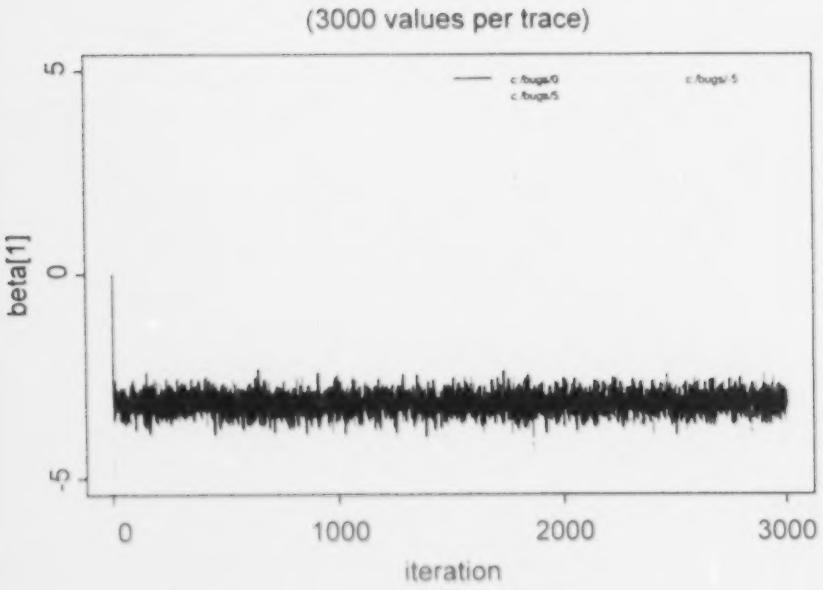
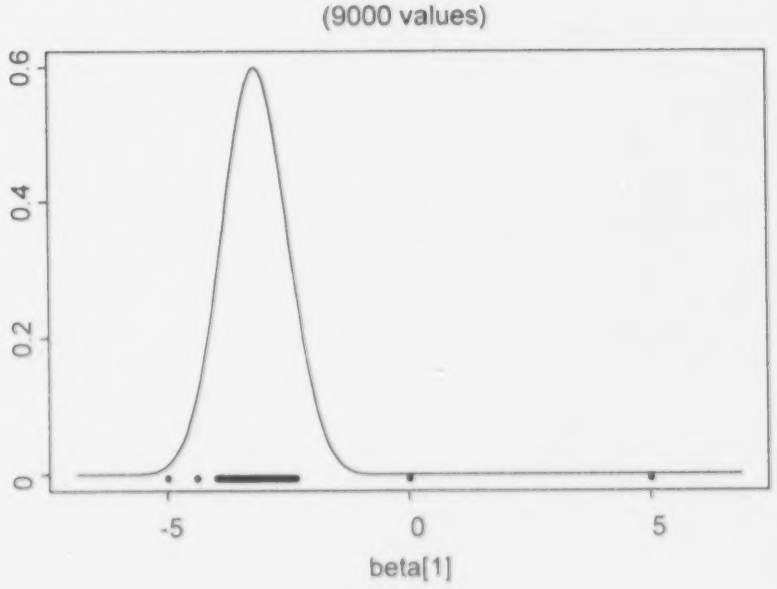
alpha
iteration


(2000 values)
alpha

Convergence with Starting Values for Usage Item-1





Appendix

```
model lsat6;
const
   I = 1000,
   J = 5;
var
   y[I,J], p[I,J], theta[I], alpha, zeta[J], b[J];
data in "lsat6-s.dat";
inits in "rasch.in";
{
  for (i in 1:I) {
     for (j in 1:J) {
        logit(p[i,j]) <- alpha*theta[i] - beta[j];
        y[i,j] ~ dbern(p[i,j]);
     }
     theta[i] ~ dnorm(0,1);
  }
  for (j in 1:J) {
     beta[j] ~ dnorm(0,0.0001);
     b[j] <- - beta[j] - mean(beta[]);
  }
  alpha ~ dnorm(0,0.0001) I(0,);
}
```

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